PHYSICS CLUB
OCTOBER STEM SCHOOL

A Students' Handbook To MECHANICS

Publications Committee

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1st Edition

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Introduction

This *Mechanics Handbook* is the product of dedicated effort by the Publications Committee of the STEM October Physics Club. Driven by the field forces of physics and education, this handbook has been crafted to serve as a concise and accessible resource for students and enthusiasts alike.

It is important to note that this handbook is not intended to replace comprehensive, calculus-based physics textbooks. Instead, it is designed as a simplified companion—a "hand" book, after all—that condenses core concepts, equations, and methodologies of mechanics into an easily digestible format. It is optimized for quick reviews and revisions.

The development of this handbook is entirely non-profit. Its use or distribution will generate no revenue. Its sole purpose is to help the growth of its readers and to encourage a love for physics among learners.

We would also like to acknowledge that some figures included in this handbook have been sourced from various physics textbooks and online sources. Due to the unavailability of specific references, we regret that we cannot attribute them to their original creators, but we acknowledge the contributions of the original authors. We do not claim these figures as our creations.

We hope this handbook proves to be a valuable tool on your educational journey. As you explore its pages, may you uncover the elegance of mechanics and find inspiration to explore the cosmos with your minds accelerating, momentum growing, and books ever at your side.

Happy learning!
Seif Mohamed
Organizer and Founder of Publications Committee
STEM October Physics Club

Acknowledgment

This *Mechanics Handbook* is the result of the collective efforts of a dedicated team of writers, led by the Publications Committee. I would like to express my gratitude to the entire team for their hard work and commitment to this project. Each unique contribution has been invaluable in bringing this handbook to life.

The following individuals played key roles in the development of this handbook:

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- Mustafa Mohammed

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Chapter 1

Physics and Measurement

Contents

1.1 Fundamental and Derived Quantities

Physics, from a very wide perspective, can be defined as the science concerning Matter, energy, Forces, and their interactions on both macroscopic and microscopic levels. Its primary objective is formulating comprehensive principles that explain the various seemingly chaotic phenomena in the observable universe. The first step taken to reach this goal is to define various physical quantities repenting distinct aspects of nature. The most commonly used categorization is Fundamental quantities and derived quantities.

> **Remark 1.1.** Length, mass, and time are considered the three fundamental quantities for mechanics

Derived quantities, as we already established, are those the can be expressed in terms of one or more Fundamental physical quantity using mathematical relationships. The list for the examples is never ending including velocity ($\frac{Length}{Time}$), acceleration ($\frac{Velocity}{Time} = \frac{Length}{Time \times Time}$), and force ($Mass \times Acceleration$).

Definition 1.1. Derived physical quantities are physical quantities that are derived from one or more fundamental physical quantities

1.2 SI Units of Length, Mass, and Time

1.2.1 SI units

Upon measuring a mass, the result could be expressed in kilograms, pounds, ounces, stones, or carats. Even this list is so far from complete. In the scientific world, standardization is often a virtue. Scientists want to say "The rod length is 3 meters" with everyone hearing this statement full aware of how long is a 3 meter rod and with no need to get out of their way for the sake of conversion to inches or feet. For this reason, in 1790, the French National Assembly announced the development of a new decimal-based system of measurement that was then known by the metric system. The system introduced standard units such as meter for length, kilogram for mass, and second for time. Over time, the metric system evolved gradually keeping up with new discoveries in physics.In 1960, the international system of units (SI) was officially established. Expanding upon the metric system, it introduced standard units for the rest of the fundamental physical quantities such as the candela for luminous intensity and the ampere for electric current.

1.2.2 Length

Length is one of the most basic and intuitive physical quantities. It is defined to be the distance or separation between two points in space. Although length's standard unit is now well known as the meter, over the centuries it underwent many changes. At one point the unit foot was adopted by the french king Louis XIV and was defined as the length of his foot. Only less than five miles apart, in England the unit yard was defined as the distance between the tip of the king's nose to the end of his fully stretched arm. It can be deducted that those two units were not on any degree of precision and were vulnerable to continuous change. The first trial to standardize a unit for length was defining a meter as one ten-millionth of the distance from the north pole through the equator along a particular longitudinal line passing through France.

> **Remark 1.2.** the equator-north pole definition is only applicable for earth and can not be applied all over the universe

However, the definition quickly changed to be the distance between two marks on a platinum–iridium stored in France under controlled conditions. To accommodate science's need for a more accurate definition, by 1970's a meter was said to be 1 650 763.73 wavelengths orange-red light emitted from a krypton-86 lamp. In October 1983, the meter was for the last time redefined using the measured speed of light.

Definition 1.2. 1 meter is defined to be the distance traveled by light in vacuum during a time interval of 1/299, 792, 458 second.

1.2.3 Mass

Mass is the fundamental quantity in physics that is often described as the amount of matter in an object. More specifically speaking, mass is the measure of an object's inertia or the property due which the object posses a gravitational field.

> **Remark 1.3.** mass is not to be confused with the quantity " amount of substance" since the amount of substance refers to the **number** of particles in an object either atoms or ions.

Kilogram, the SI unit of mass, is the only unit that is still anchored to a physical object. A kilogram is defined to be the mass of a specific platinum–iridium alloy cylinder called "Le grand K" stored at the International Bureau of Weights and Measures at Sèvres, France. The platinum-iridium alloy was chosen because its extreme degree of stability and tendency to not lose any considerable amount of its mass. Duplicates of the alloy were made and distributed to various countries around the world. Le grand K has been able to behold the definition of a kilogram up until this day even though some scientists recently have been demanding to redefine the kilogram using a perfect silicon sphere since new technologies have allowed us to have an exact count of the atoms in such spheres Thus, a very accurate measure of its mass.

1.2.4 Time

Time is the measure of the progression of events from the past through the present to the future. It is a continuous and unidirectional quantity. Time is unavoidable when dealing with any dynamic system. It is present from the simplest equations of motion all the way to the complex interactions in quantum mechanics. The standard unit of time was firstly defined using Earth's mean solar day, the time interval between successive appearances of the sun at the same point in the sky, to be such that 1 second is $(\frac{1}{60})\cdot(\frac{1}{60})\cdot(\frac{1}{24})$ of one solar day. however, the same problem that with the first definition equator-north pole definition of length that is being dependant on Earths characteristics and not a standard universal definition.

With new technological advances, it was possible to redefine the second in a more accurate way using the atomic clocks which is able to measure the vibrations of cesium atoms. One second up until now is defined to be 9 192 631 770 times the period of radiation from the cesium-133 atom.

1.3 Dimensional Analysis

Even though they are mostly mistaken by just the special dimensions, dimensions in physics are far more broad. A dimension is a measure of a physical quantity representing its nature. They provide a way to quantify and understand the relationships between physical quantities. Each fundamental physical quantity has its fundamental dimension. For example, the dimension for length is L, for mass it is M, and for time it is T. Similarly, derived quantities has their own derived dimension that is derived from the fundamental dimensions, For example, the dimension for velocity is LT^{-1}, for acceleration it is LT^{-2}, and for force it is MLT^{-2}.

Remark 1.4. Usually the dimension of a quantity is donated using the brackets [ ]. For example you might come a cross the equation $[v] = LT^{-1}$ which means that the dimension of the quantity v (velocity) is LT^{-1}

Remark 1.5. dimensions are independent of the unit used to measure them. Length could be measured in meter, inch, or feet. and have the same dimension of L

Dimension analysis is an essential tool for scientists and engineers. It involves studying the dimensions of various quantities within an equation to check its consistency and derive relationships. It is based on two key principles the first being that any quantity can be expressed in terms of the seven fundamental quantities and their respective dimensions, and the second is known as dimensional homogeneity that helps in verifying the correctness of physical equations and ensures that they are dimensionally consistent.

Definition 1.3. Dimensional Homogeneity means that for any physical equation to be valid

1. Both sides of the equation must have the same dimensions
2. Any two quantities related with a summation or subtraction relation must have the same dimensions

Example 1.1

Check the dimensional consistency of the equation for kinetic energy $KE = \frac{1}{2}mv^2$

Solution. First of all, we acquire the dimension for each quantity in the relationship.
$Mass = M$
$Velocity = LT^{-1}$
$Energy = ML^2T^{-2}$
Now we replace each quantity in the equation with its respective dimension disregarding any constants
$ML^2T^{-2} = M \cdot (LT^{-1})^2 = M \cdot L^2T^{-2} = ML^2T^{-2}$

■

We notice that both sides of the equation match, thus the equation is dimensionally consistent and MAYBE true

Remark 1.6. Dimensional analysis disregard any constant multipliers, thus a dimensionally consistent equation is not necessarily a correct equation and can not be used to get specific values such as the case in the equation for kinetic energy

Dimensional analysis significance is far more beyond just a checking tool. In the wider scoops of engineering and advanced physics, dimensional analysis plays a crucial role in simplifying complex problems. In fluid dynamics, for example, a common procedure is to reduce the variables in any equation by breaking down each quantity into its basic MLT dimension form, combine the dimensions from different variables in a way that makes them cancel each other, then this term is replaced by dimensionless numbers such as Reynolds number, Fronde number, and Mach number. A crucial tool in that process is Buckingham π Theorem.

Definition 1.4. Buckingham π Theorem states that if a physical problem involves n variables and these variables contain k fundamental dimensions, the problem can be reduced to $n - k$ dimensionless groups.

This strategy has contributed positively to scientific research. Reducing the variables is invaluable to researchers in fields such as fluid dynamics or heat transfer as it makes the problems and simulations more traceable by allowing the researchers to just focus on the dimensionless numbers. As well as providing a method for checking the consistency of equations, validating their claims, and identifying potential errors.

1.4 Significant Figures

significant figures, also known as significant digits, are the digits in a number that hold meaningful information about the precision of the measure. They are used within the scientific community to communicate the accuracy of a measure and ensure consistency in calculations. To determine how to get the number of significant figures, several rules are followed

Remark 1.7.

1. **Non-zero digits,** Any digit that is not a zero, are always significant.
2. **Leading zeros,** Zeros which precede all non-zero digits, are not significant.
3. **Captive zeros,** Zeros that are between non-zero digits, are significant.
4. **Trailing Zeros,** Zeros after which no other non-zero digits follow, are not significant unless there is a decimal point.
5. **Exact numbers,** Numbers resulting from counting or a defined quantity, have infinite significant figures.

Example 1.2

Determine the number of significant figures in (a) 00.0453, (b) 1000, (c) 7.890, (d) 0.03020

Solution. (a): 3 significant numbers, the leading zeros are not significant, and digits from 4 to 3 are significant.
(b): 1 significant figure, the digit 1 is significant, and the trailing zeros are not significant since there is no decimal point
(c): 4 significant figures, the digits from 7 to 9 are significant and the trailing zero is significant since there is a decimal point
(d): 4 significant figures, the digits 3 and 2 are significant, the captive zero between 3 and 2 is significant, and the trailing zero is significant

■

Remark 1.8. When performing calculations, the number of significant figures must be considered to ensure precision and accuracy:

1. **Addition and Subtraction,** The result should have the same number of decimal places as the measurement with the fewest decimal places.
2. **Multiplication and Division,** The result should have the same number of significant figures as the measurement with the fewest significant figures.

Example 1.3

Perform the following operations with correct significant figures:

1. **Addition:** $12.11 + 3.2$
2. **Subtraction:** $15.45 - 0.134$
3. **Multiplication:**2.34×1.2
4. **Division:** $7.89/1.1$

Solution. 1. **Addition:**

$$12.11 + 3.2 = 15.31 \approx 15.3$$

The result should have the same number of decimal places as the measurement with the fewest decimal places (1 decimal place).

2. **Subtraction:**

$$15.45 - 0.134 = 15.316 \approx 15.32$$

The result should have the same number of decimal places as the measurement with the fewest decimal places (2 decimal places).

3. **Multiplication:**

$$2.34 \times 1.2 = 2.808 \approx 2.8$$

The result should have the same number of significant figures as the measurement with the fewest significant figures (2 significant figures).

4. **Division:**

$$7.89/1.1 = 7.172727 \approx 7.2$$

The result should have the same number of significant figures as the measurement with the fewest significant figures (2 significant figures).

■

Chapter 2

Motion in One Dimension

Contents

2.1 Displacement, Velocity, and Acceleration

Motion can occur in multiple dimensions, starting with one and up to three. Understanding motion in our universe seems to be an easy yet essential task; consequently, defining some key concepts is a crucial step in the journey of comprehending the kinematics of objects, starting with the motion in only one dimension.

2.1.1 Displacement

You may have heard of that term before, but can you precisely define it? Perhaps. Generally, displacement is a physical quantity defined as the *change* in an object's *position*. Suppose that an object's position is defined at some point in the rectangular coordinate system. If, at any time, the object changed its position with respect to its initial one, it is said to have done displacement, and the displacement can be calculated through simply subtracting the initial position coordinates from the final ones (i.e. $x_f - x_i$, where x_f is the final position and x_i is the initial one). Displacement is denoted by Δx.

Example 2.1

A particle's position is set to be 3 m away with respect to some reference of frame. Supposing that the particle changed its position and is now defined to be as 11 m away with respect to the same reference. Calculate the displacement done by the particle.

Solution. Referring back to the displacement definition, the displacement done by the car can be calculated as follows:

$$\Delta x = x_f - x_i = 11 - 3 = 8m$$

Note that the resulting value represents the *change* in the position and NOT the new position. ■

2.1.2 Velocity

Having discussed what displacement is, let's now move on to define other related quantities as well. Suppose that a particle has done displacement, which is simply changing its position, over a certain time interval. The amount of change of its position over the time period is called the rate of change of an object's position, which you may know as *velocity* (denoted as v). That is, velocity is the rate of change of the displacement of an object. For example, if a particle is doing a certain displacement in a specific time period, it is said to be having an average velocity of the value resulted from dividing the value of the total displacement by the value of the total time interval.

Example 2.2

A particle's position is defined to be at 7 m at t = 0. After the elapse of 10 seconds, the particle's new position is now at 117 m from the origin. Find the velocity of the particle.

Solution. From the definition, velocity is the rate of change of displacement and it is calculated as follows:

$$v = \frac{\Delta x}{\Delta t} = \frac{117 - 7}{10 - 0} = \frac{110}{10} = 11m/s$$

■

Involving methods of calculus, we could define velocity as the derivative of a displacement function with respect to time. Derivatives are the instantaneous rate of change of a particular function with respect to a specific variable.

$$v = \frac{dx}{dt}$$

Example 2.3

A particle's position is defined to be a function of time written as following $x(t) = 2t^3 + 4t$. Find the velocity of the particle at $t = 5s$.

Solution. This is a pretty straightforward example to apply the definition on the numeric level. Using the power rule, the velocity should be executed as follows:

$$v(t) = \frac{d}{dt}x(t) = 2 \cdot 3t^{3-1} + 4 \cdot 1t^{1-1} = 6t^2 + 4$$

Having obtained the velocity function, we can calculate at any given instant. As per the question, we could work it out for $t = 5s$ by simply inserting 5 whenever we encounter a t.

$$v(5) = \frac{dx}{dt}_{t=5} = 6(5)^2 + 4 = 154m/s$$

■

2.1.3 Acceleration

You may have expected what acceleration is, and you are probably right. Analogously speaking, if an object's velocity is changing over a time period, it's said to be undergoing acceleration. In other words, acceleration is the rate of change of velocity. Imagine a particle is doing displacement over a time period and therefore having velocity. If the rate of change of displacement is changing (i.e. the velocity is changing), so this object does have acceleration.

Example 2.4

As it starts at a velocity of $4m/s$, the particle sped up to $16m/s$ in $4s$. Find the particle's acceleration.

Solution. You are probably fed up from that particle, but you would encounter more interesting examples as we go into higher dimensions, promise. As per the definition, the acceleration is the rate of change of velocity; therefore, it should be computed as follows:

$$a(t) = \frac{\Delta v}{\Delta t} = \frac{16 - 4}{4} = 2m/s^2$$

■

Definition 2.1. Calculusly speaking, as the **acceleration** is the rate of change of the rate of change of displacement, it happens to be the second-order derivative of the displacement as well as the first-order derivative of the velocity.

(Yes you are right *Calculusly* is not an actual word, but it is descriptive enough to match the deep physics passion of the authors)

$$a(t) = \frac{dv}{dt}$$

Example 2.5

A particle's position is defined to be a function of time written as following $x(t) = 4t^2 - 3t^3$. Find the acceleration of the particle at $t = 6s$

Solution. This example requires another step of calculus computations but it's still straightforward application of the definition. As the acceleration is the second-order derivative of the displacement function with respect to time. That is, we will work out the acceleration through simply differentiating the displacement function twice.

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 8 - 18t$$

$$a(6) = \frac{d^2x}{dt^2}_{t=6s} = 8 - 18(6) = -100m/s^2$$

■

You might be wondering what is the meaning of the negative acceleration? or whether if it's even possible to be? The negative sign simply means that the acceleration is just being in a direction opposite to the direction of motion. And every acceleration, and similar quantities, needs to be defined by both the magnitude and direction–concisely speaking because they are ***vectors***, which are our next chapter topic.

2.2 Kinematic Equations

When a particle's velocity is changing on a regular basis, we say that the particle is undergoing constant acceleration. This type of motion is very common. Thus, physicists formulated mathematical equations to describe and analyze such models. Please note that all of these equations would only works with objects moving with ***constant*** acceleration and are assuming that the terms indicate the following quantities only in the x-dimension.
Moving on now to discuss those kinematic equations, we would start with the most basic one. As previously mentioned,

$$a = \frac{\Delta v}{\Delta t}$$

and if we deconstructed this simple equation into more elementary components, it would be as follows:

$$a = \frac{v_f - v_i}{t}$$

$$a \cdot t = v_f - v_i$$

$$v_f = v_i + a \cdot t \tag{2.1}$$

Additionally, as per our discussion on motion with constant acceleration, velocity varies linearly since the value of the rate of change (i.e. acceleration) is constant. Consequently, we could express **average** velocity in the following mathematical form:

$$v_{avg} = \frac{v_f + v_i}{2}$$

Furthermore, the second kinematic equation is to compute displacement for an object undergoing constant acceleration. Assuming that $v = \frac{x_f - x_i}{t}$ is the average velocity, so along with $v_{avg} = \frac{v_f - v_i}{2}$, the following is obtained:

$$x_f - x_i = v \cdot t = \frac{1}{2}(v_f + v_i) \cdot t$$

$$x_f = x_i + \frac{1}{2}(v_f + v_i) \cdot t \tag{2.2}$$

Substituting the v_f with 2.1 and applying the distribution property of multiplication, we proceed as follows:

$$x_f = x_i + \frac{1}{2}((v_i + at) + v_i) \cdot t$$

$$x_f = x_i + \frac{1}{2} \cdot 2 \cdot v_i \cdot t + at \cdot t$$

$$x_f = x_i + v_i \cdot t + \frac{1}{2}a \cdot t^2 \tag{2.3}$$

Back to equation 2.2, we can obtain a time variable-free equation through substituting for time using 2.1

$$x_f = x_i + \frac{1}{2}(v_f + v_i)(\frac{v_f - v_i}{a})$$

$$x_f = x_i + \frac{v_f^2 - v_i^2}{2a}$$

$$x_f - x_i = \frac{v_f^2 - v_i^2}{2a}$$
$$v_f^2 = v_i^2 + 2a(x_f - x_i) \tag{2.4}$$

2.2.1 Again, But Calculusly Speaking

In this section, we are just deriving the same kinematic equations but with a calculus-based approach. Recall that we defined acceleration as the rate of change of velocity. Translating that to the language of calculus, it would be as follows:

$$a = \frac{dv}{dt}$$

For pure mathematicians, what we are going to do is a pure act of crime to treat a derivative as a *"fraction"* and we should be sentenced to the court of justice. However, as physicists, we agree upon accepting this only for the matter of simplification.

$$dv = adt$$
$$\int_{v_i}^{v_f} dv = a\int_0^t dt$$
$$v_f - v_i = a(t - 0)$$
$$v_f = v_i + a \cdot t \tag{2.5}$$

Also, we defined velocity as the rate of change of displacement. With the same analogy, we would proceed as follows:

$$v = \frac{dx}{dt}$$
$$dx = vdt$$
$$\int_{x_i}^{x_f} dx = \int_0^t vdt$$

So what is different here is that v is not a constant value as a. What we need to do for instance is replace the v with its function of t, or 2.1, so that we could integrate with respect to dt

$$x_f - x_i = \int_0^t v_i + a \cdot tdt$$

$$x_f - x_i = \int_0^t v_i dt + \int_0^t a \cdot t dt$$

$$x_f - x_i = v_i \cdot (t - 0) + a \cdot \frac{t^{1+1}}{1+1}$$

$$x_f = x_i + v_i t + \frac{1}{2} a t^2 \qquad (2.6)$$

2.3 Free Fall

Free falling is primarily the motion of an object in the absence of air resistance under the influence of the force of gravity. Although it may sound complicated, it's a pretty easy situation to imagine–actually to perform! Grab a coin and a piece of paper, and drop them from a certain height. You would notice that the coin would reach faster than the piece of paper. But why that happen although they are under the influence of the same force?

The answer is *air resistance*. That is, the air resistance gets greater as much as the surface area of the freely falling object. Consequently, air resistance affects the falling of the paper and hinders it from being on the same calibre as the coin. However, if we over folded that piece of paper to be small enough, it would be able to reach in an apparently the same amount of time as the coin does. And as we ignore air resistance in our analysis models for instance, both objects should be going under the same constant acceleration, also known as g, caused by the gravitational field of the Earth. On Earth, g is assessed to be equivalent to $9.81 m/s^2$. Nevertheless, it varies from an object to another, and maybe even from a place to another, which is going to be discussed in detail in chapter 13.
For the sake of simpler calculations, sometimes we round the *the acceleration due to gravity* to be $10 m/s^2$. Additionally, freely falling objects do not have to be starting their initial motion from rest, in analysis models.

Example 2.6

Aslam and Atef decided to play football after a long physics study session. They love to do long passes as they are both tall and fast, that's why they surpass everybody else on the pitch. However, Aslam went insane and struck the ball so that it went vertically upwards instead of doing a horizontal long pass. The ball's initial velocity is $4m/s$. Find
(a) The vertical distance, given that it took the ball 1.53 seconds to reach the maximum height without the presence of air resistance.
(b) The final velocity at the instant right before hitting the ground.

Solution. (a) To solve this question, we would look into which kinematic equation we need based on the variable we have and the one we are finding.

- $x \Rightarrow$ Wanted
- $v_i \Rightarrow$ Given
- $t \Rightarrow$ Given
- $a \Rightarrow g$ (The acceleration due to gravity since this is free fall)

$$x_f = x_i + v_i t + \frac{1}{2}at^2$$

$$x_f = 0 + 4 \cdot 1.53 \cdot 2 + \frac{1}{2} \cdot 9.81 \cdot (1.53 \times 2)^2 = 58.2m$$

We multiplied t by 2 since the given value is for half the trip only!
(b) Using the same strategy,

- $v_f \Rightarrow$ Wanted
- $v_i \Rightarrow$ Given
- $t \Rightarrow$ Neither given nor wanted

Consequently, we will take the kinematic equation that doesn't contain time as a term since they are only two equations for finding the final velocity.

$$v_f^2 = v_i^2 + 2a\Delta x$$

$$v_f = \sqrt{v_i^2 + 2a\Delta x}$$

$$v_f = \sqrt{4^2 + 2 \cdot 9.81 \cdot 58.2} = 34m/s$$

■

Chapter 3

Vectors

Contents

Vectors are considered the cornerstone in defining a lot of quantities in physics and even in our daily lives. In this chapter, we will study the positions of objects or points in two- or three-dimensional space that require two types of information: distance from a reference point and direction relative to a reference axis.

Definition 3.1. Vectors are physical quantities that need both a magnitude and a direction to be defined, like displacement, velocity, acceleration, ... etc.

Definition 3.2. Scalars are physical quantities that require just a magnitude to be defined, like mass, time, temperature, ... etc.

Remark 3.1. The magnitude does represent both a **number** and a **measuring unit**.

3.1 Vector Forms

We can represent vectors in many ways. The most common forms are:

1. Component form (Rectangular)
2. Unit vector form
3. Magnitude and direction form (Polar)

3.1.1 Component form

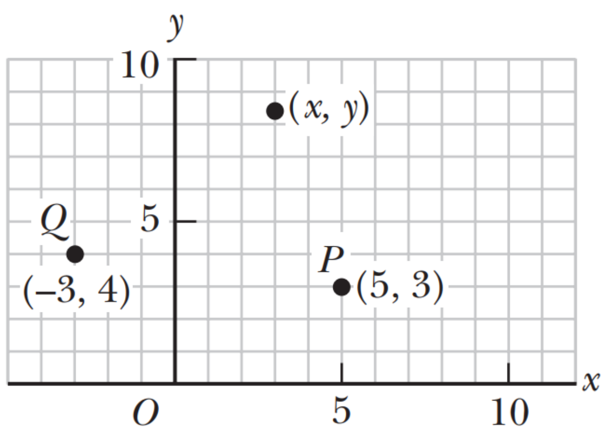


Figure 3.1: Coordinate system

This form utilizes the Cartesian coordinate system as shown in Figure 3.1. In two dimensions, Cartesian coordinates of a point in space represent the x and y values of the point, and the vector is expressed as (x, y). In three dimensions, it will be, for example, (x, y, z) and so on.

3.1.2 Unit Vector form

A unit vector is one whose magnitude is equal to one. The "cap" symbol $(\hat{x})$ is used to indicate unit vectors. In other words, When any vector is divided by its magnitude (norm), this will give us a unit vector.

$$\frac{\vec{A}}{|A|} = \hat{A} \tag{3.1}$$

The way we can represent a vector using the unit vector form depends on our definition of the reference axes. Standard axes are perpendicular x, y axes. The unit vector in the direction of the x axis is denoted by $(\hat{\imath})$ and on the y axis is denoted by $(\hat{\jmath})$. So, for instance,

$$\vec{A}(x, y) = x\hat{\imath} + y\hat{\jmath} \tag{3.2}$$

3.1.3 Polar form

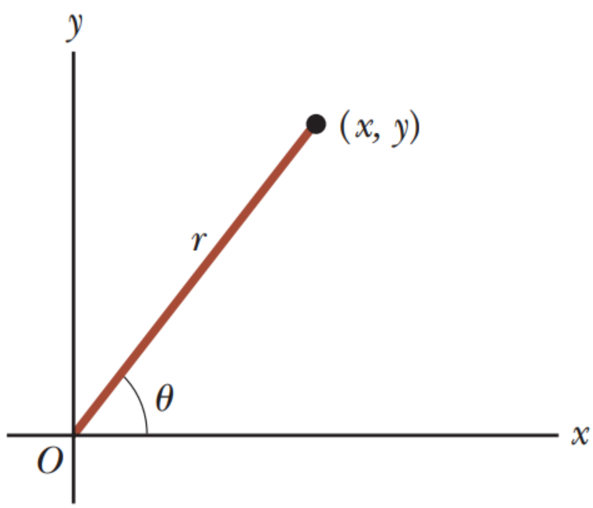


Figure 3.2: Polar Form

In a polar coordinate system, a vector is represented using (r, θ), where r is the distance from the origin to the point having Cartesian coordinates (x, y), (i.e., the norm of the vector) and θ is the angle between a fixed axis and a line drawn from the origin to the point as shown in Figure 3.2. The fixed axis is often the positive x-axis, and θ is usually measured counterclockwise from it.

Now, if we have the component form of the vector, then we can convert it into the polar form, and vice versa. The way that relates the two forms with each other is our utilization of trigonometry. Let's see an example.

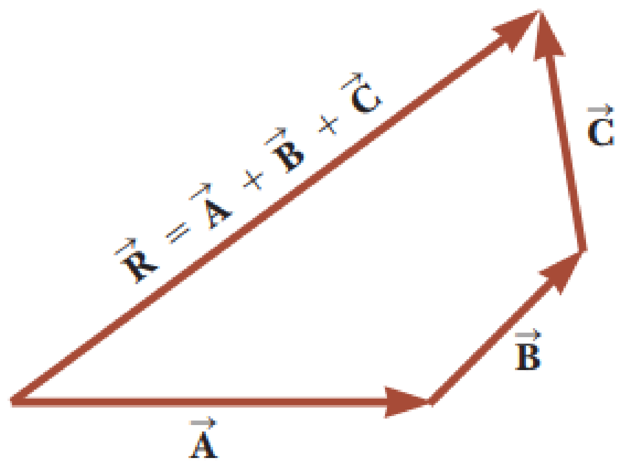


Figure 3.3: Geometric construction for vectors addition

Example 3.1

$\vec{T}$ has a Cartesian coordinates $(x, y) = (-3.50, -2.50)$ m. Re-form $\vec{T}$ to be represented in **polar form**.

Solution. In polar form, we need two things, r and θ From Pythagorean identity:

$$r^2 = x^2 + y^2 \tag{3.3}$$

So, $r = \sqrt{x^2 + y^2} = \sqrt{(-3.50)^2 + (-2.50)^2} = 4.30$ m

Now we can use any of the three identities to get the angle θ

$$\implies \tan\theta = \frac{y}{x} = \frac{-2.50 \text{ m}}{-3.50 \text{ m}} = 0.714$$

$$\theta = \arctan(0.714) = 216^\circ$$

Then, $\vec{T}(-3.50, -2.50)$ or $\vec{T}(4.30 \text{ m}, 216^\circ)$ ■

3.2 Vector Addition

3.2.1 Graphically

Vector addition can be thought of graphically. When a vector $\vec{A}$ is added to another $\vec{B}$, the two vectors can be reorganized in order to connect the head of one vector to the tail of the other, and the vector that points from the tail of the first vector to the head of the second vector will be the **Resultant** of these two vectors (i.e, $\vec{R} = \vec{A} + \vec{B}$). If there is more than two forces, the same steps apply as shown in Figure 3.3. This technique is called "Head to tail method".

Remark 3.2.

1. $\vec{A}+\vec{B}=\vec{B}+\vec{A}$
2. $\vec{A}+\left(\vec{B}+\vec{C}\right)=\vec{C}+\left(\vec{A}+\vec{B}\right)$

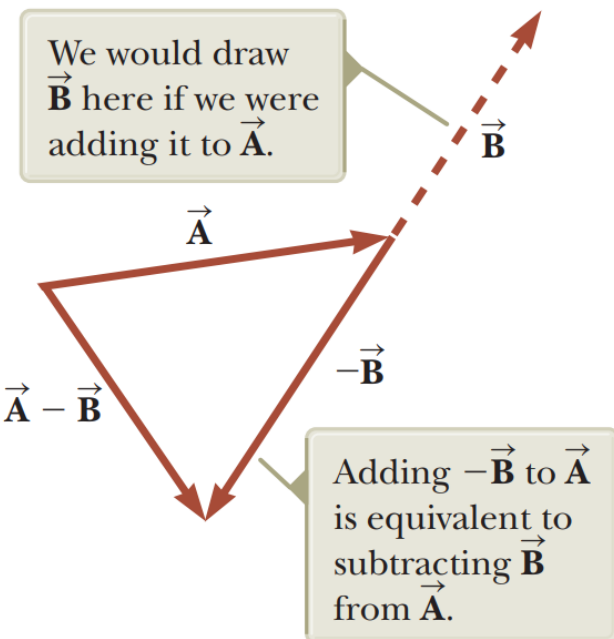


Figure 3.4: Subtraction of Vectors

The operation of vector subtraction makes use of the definition of the negative of a vector. The negative of the vector $\vec{A}$ is defined as the vector when added to $\vec{A}$ gives zero for vector sum. The vectors $\vec{A}$ and $\vec{-A}$ have the same magnitude but in opposite directions. So, $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ as illustrated in 3.4.

3.2.2 Algebraically

After we were introduced to the graphical representation of vectors, we need to know how to add vectors algebraically without the need for graphical representation each time.
For example, we have two arbitrary vectors that make an angle $(\widehat{AOB})$ with

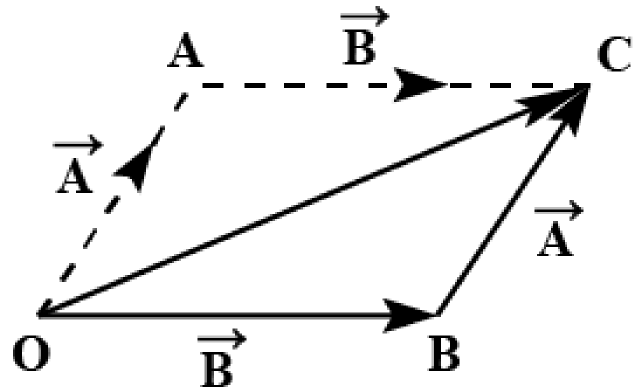


Figure 3.5: Algebraic Vector Addition

each other as shown in Figure 3.5. Let $(\widehat{AOB})$ be θ. Then ,$(\widehat{OBC})$ equals $(\pi - \theta)$.

By Cosine law of triangles, if we have the magnitudes of $\vec{A}$ & $\vec{B}$ and the value of the angle subtended by them, we can easily solve for their Resultant $\vec{C}$

$$\begin{aligned} \mathbf{C} &= \sqrt{\mathbf{A}^2 + \mathbf{B}^2 - 2 \cdot \mathbf{A} \cdot \mathbf{B} \cdot \cos(\pi - \theta)} \\ &= \sqrt{\mathbf{A}^2 + \mathbf{B}^2 + 2 \cdot \mathbf{A} \cdot \mathbf{B} \cdot \cos(\theta)} \end{aligned} \tag{3.4}$$

Example 3.2

Consider two vectors, **A** and **B**, in two-dimensional space with magnitudes $|\mathbf{A}| = 5$ and $|\mathbf{B}| = 3$. The angle between **A** and the x-axis is $\theta_1 = 60°$, and the angle between **B** and the x-axis is $\theta_2 = 45°$. Find the magnitude of vector **C** obtained by adding **A** and **B**.

Solution. Using the cosine law, we can write the equation for the magnitude of vector **C** as:

$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2 \cdot |\mathbf{A}| \cdot |\mathbf{B}| \cdot \cos(\theta)$$

Substituting the given values, we have:

$$|\mathbf{C}|^2 = 5^2 + 3^2 - 2 \cdot (5) \cdot (3) \cdot \cos(\theta)$$

Now, we need to determine the angle θ between vectors **A** and **B**. From the given information, we know that the angle between **A** and the x-axis is $\theta_1 = 60°$, and the angle between **B** and the x-axis is $\theta_2 = 45°$.

To find the angle θ between **A** and **B**, we can subtract the two angles: $\theta = \theta_2 - \theta_1 = 45° - 60° = -15°$

Note that we take the negative value since θ_1 and θ_2 are measured counterclockwise from the x-axis. However, taking the positive value will yield the same result, as the cosine function doesn't change from the first to fourth quads.

Now, substituting the values into the equation:

$$|\mathbf{C}| = \sqrt{25 + 9 + 2 \cdot (5) \cdot (3) \cdot \cos(-15°)} \approx \sqrt{62.98} \approx 7.94$$

■

There is another solution!!

Solution. Since we have each vector's magnitude and the angle between them and the x-axis, we can use the two trigonometric identities:

$$A_x = A\cos\theta \tag{3.5}$$
$$A_y = A\sin\theta \tag{3.6}$$

The same identity applies to **B**, so:

$$A_x = 5 \times \cos 60 = \frac{5}{2} \qquad A_y = 5 \times \sin 60 = \frac{5\sqrt{3}}{2}$$
$$B_x = 3 \times \cos 45 = \frac{3\sqrt{2}}{2} \qquad B_y = 3 \times \sin 45 = \frac{3\sqrt{2}}{2}$$

Adding $A_x$toB_x, we get the total x component for the two vectors, and the same applies

$$C_x = A_x + B_x \tag{3.7}$$
$$C_y = A_y + B_y \tag{3.8}$$

So,

$$C_x = \frac{5}{2} + \frac{3\sqrt{2}}{2} = \frac{5+3\sqrt{2}}{2}$$
$$C_y = \frac{5\sqrt{3}}{2} + \frac{3\sqrt{2}}{2} = \frac{5\sqrt{3}+3\sqrt{2}}{2}$$

We can represent $\vec{C} = \frac{5+3\sqrt{2}}{2}\hat{i} + \frac{5\sqrt{3}+3\sqrt{2}}{2}\hat{j}$
Now we can conclude the magnitude of the resultant since we have the values of its $x\&y$ components:

$$\begin{aligned} C &= \sqrt{C_x^2 + C_y^2} \\ &= \sqrt{\left(\frac{5+3\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{3}+3\sqrt{2}}{2}\right)^2} \\ &= 7.94 \end{aligned}$$

■

Remark 3.3. The previous solution utilizes the concept of **Resolution of vectors**. This technique is a valid method for vector addition, as depicted in figure 3.6.

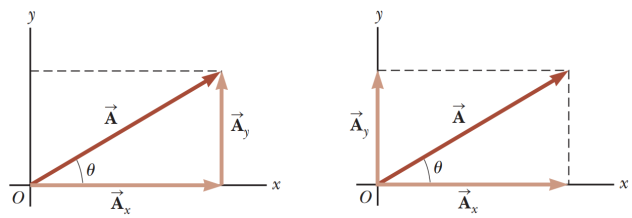


Figure 3.6: Vector $\vec{A}$ can be represented as a vector sum of its components $\vec{A}_x$ and $\vec{A}_y$

3.3 Vector Multiplication

3.3.1 Scalar Product

Scalar or Dot product can be thought in terms of **Directional Multiplication**. That is, all the multiplications in the same direction count. Let's clarify it a bit more.

When we do 3×4, it is clear that the product is 12. However, we can think of it in another way. Doing 3×4 is just like we are **scaling** growth rates. In other words, we have our 3x growth and make it 4x as large to get 12x. These are straightforward growth rates when we deal with just scalars (numbers), but what about vectors? Well, the main obvious difference in vectors is that we are dealing with **directional growth**. Suppose we treat 3×4 as a dot product:

$$(3, 0) \cdot (4, 0)$$

The number 3 is "directional growth" in a single dimension (the positive x-axis, let's say), and 4 is "directional growth" in that same direction. 3 x 4 = 12 means we get 12x growth in a single dimension.

Now, suppose 3 and 4 refer to different dimensions:

$$(3, 0) \cdot (0, 4)$$

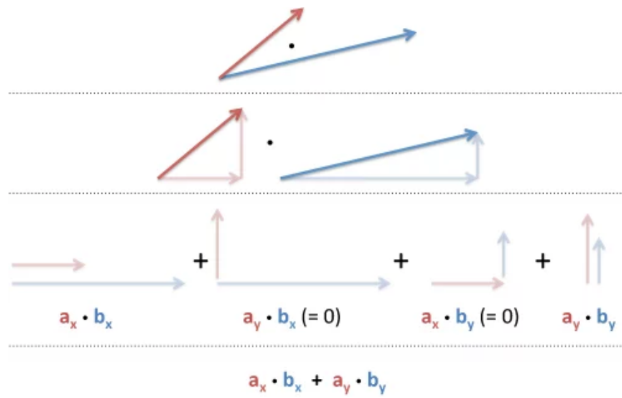


Figure 3.7: Dot product: piece by piece

So, the growth rate in the first vector is growing in the east direction, but the second grows in the north direction, so there is not any overlap between the growth rates of the two vectors. (i.e., $(3,0)\cdot(0,4)=0$). That is what we mean "all the multiplications in the same direction count", and we just add the results if we have more than one directional growth. Now, let's look at the exact calculation method so that we can come up with the formula for the dot product. We have two methods: the first one is as shown in Figure 3.7. We first break down each vector into its $x-y$ components, then we will do **FOIL** as if they were two brackets. As we mentioned before, the dot product looks for the same component overlap. So, the perpendicular components will not count because they are in totally different directions. So, we are left with $a_x \cdot b_x + a_y \cdot b_y$, and this is one of the general formulae used for evaluating dot products.

Claim 3.1 — Consider two arbitrary vectors of n-order: $\vec{A}$ and $\vec{B}$. $\vec{A} = (a_x, a_y, ..., a_n)$ and $\vec{B} = (b_x, b_y, ..., b_n)$. Then,

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (a_x \cdot b_x) + (a_y \cdot b_y) + ... + (a_n \cdot b_n)\\ &= \sum_{i=1}^{n}(a_i \cdot b_i) \qquad (3.9)\end{aligned}$$

For the second approach, it will follow the same basis in Figure 3.7, but we will rotate the two vectors together, so one of them will coincide with x-axis

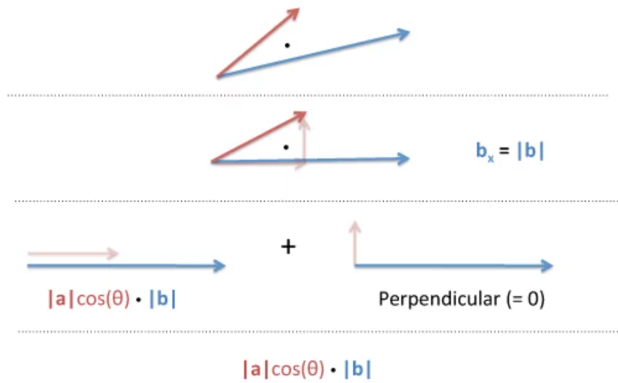


Figure 3.8: Dot product: Rotate to base line

as shown in Figure 3.8, so vector b doesn't have a y-component. We break down only vector a, and the perpendicular ones will cancel and we are left with a component on x axis multiplied by the total length of vector b. From equation (5), we can say that this component will be $a\cos\theta$. So, we have our second formula for dot product between any two vectors.

Claim 3.2 — Consider two arbitrary vectors of n-order: $\vec{A}$ and $\vec{B}$ and have an angle θ between their tails, so:

$$\vec{A}\cdot\vec{B} = \|A\|\|B\|\cos\theta \tag{3.10}$$

3.3.2 Projections

As stated above, the dot product gives us a way to measure how similar two vectors are. The problem with the dot product, though, is that it spits out a number. Sometimes we want a way to measure how well vectors travel together while still preserving some information about direction. In other words, we want a dot-product-like measurement that returns the same information as a vector rather than a scalar. How should we do this? Well, given vectors a and b, the quantity $a\cdot b$ measures how well they travel together. We could rephrase this, use a as our "reference direction" and say that $a\cdot b$ measures how well b travels in the direction of a. Since we want to preserve information about the direction we're travelling in, we can just multiply $a\cdot b$ by the vector a:

$$(a\cdot b)a$$

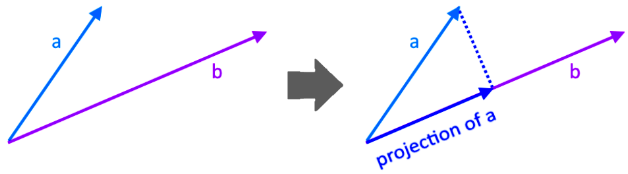


Figure 3.9: Projection

The only issue here is that the length of a is going to mess up our measurement, so to be safe, we should instead multiply the dot product by a unit vector in the a direction:

$$(a \cdot b)\frac{a}{\|a\|}$$

We could improve on one more thing. Since a is our reference direction, we (again) don't want the length of a messing up our measurements. So we could normalize the coefficient of our vector by dividing once more by the length of a:

$$\frac{a \cdot b}{\|a\|}\frac{a}{\|a\|} = \frac{a \cdot b}{\|a\|^2}a = \frac{a \cdot b}{a \cdot a}a = \frac{a \cdot b}{\|a\|}\hat{a} \tag{3.11}$$

The previous vector produced is called the **projection of b onto a**, *i.e,* $Proj_a b$

Claim 3.3 —

$$\text{Proj}_a b = \frac{a \cdot b}{\|a\|}\hat{a} \tag{3.12}$$

$$\text{Proj}_b a = \frac{a \cdot b}{\|b\|}\hat{b} \tag{3.13}$$

It turns out that this is a very useful construction. For example, projections give us a way to make orthogonal things (Perpendicular). By the nature of "projecting" vectors, if we connect the endpoints of a with its projection $\text{Proj}_b a$, we get a vector orthogonal to our reference direction a that represents the dashed line as shown Figure 3.9. In other words, the vector $a - \text{Proj}_b a$ is orthogonal to b.

3.3.3 Vector Product

Vector product or cross product — as the name implies — is an operator for vectors multiplication, but it produces a vector rather than a scalar as in the case of the dot product. In dot product, our mission was to measure how similar two vectors are. So, as predicted, cross product is the way that measures how **different** two vectors are. In other words, it measures the **orthogonality** of two vectors. Another important usage for vector product its importance in finding the **area of parallelograms**. We know that the area of the parallelogram is just the base multiplied by its height. If we have the lengths of the two adjacent sides of a parallelogram, can we find its area ? The answer is: YES, using simple trigonometry or **Cross Product**.

As shown in Figure 3.10, a and b are two vectors, and they are pointing in different directions. To find the area of the parallelogram enclosed by these two vectors, we should find the height, which is $\|a\| \sin\theta$. So, the area equals $\|b\|\|a\| \sin\theta$. Simultaneously, the magnitude of the cross product between the two vectors will give us the same area of the parallelogram produced by the two vectors!

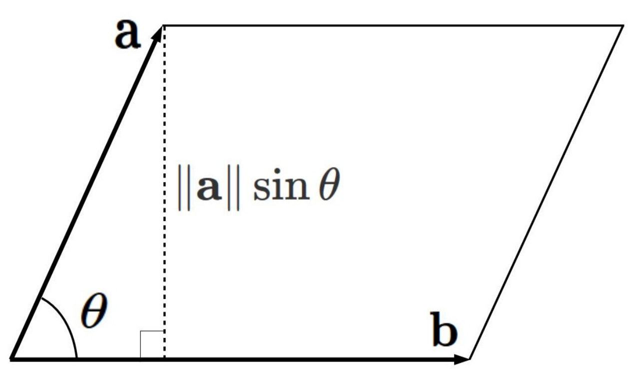


Figure 3.10: Area of parallelogram

Claim 3.4 — Consider two arbitrary vectors: $\vec{A}$ and $\vec{B}$, and have an angle θ between their tails, so:

$$|\vec{A} \times \vec{B}| = \|A\|\|B\| \sin\theta \tag{3.14}$$

Cross products were first used for their geometric applications. Algebraically, we will use the concept of determinants to evaluate the cross product of vectors, as we will see in the next example.

Example 3.3

Two vectors $\vec{F}\&\vec{T}$ have components (5, -2, 4) &(1, -7,6) respectively. Find $\vec{F} \times \vec{T}$.

Solution. To evaluate the cross product between vectors from components, we will make use of 3×3 determinant, a very simple calculation that we will stick to.

First, we will put the two vectors as illustrated below:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -2 & 4 \\ 1 & -7 & 6 \end{vmatrix}$$

We need three components x, y, z for the produced vector. So, we evaluate the determinant by putting our unit vectors as following:

$$\begin{vmatrix} -2 & 4 \\ -7 & 6 \end{vmatrix} \hat{i} + (-1) \begin{vmatrix} 5 & 4 \\ 1 & 6 \end{vmatrix} \hat{j} + \begin{vmatrix} 5 & -2 \\ 1 & -7 \end{vmatrix} \hat{k}$$

Notice that the matrix next to $\hat{i}$ comes from cancellation of the row of the unit vectors, and the column under $\hat{i}$, so we get the remaining four numbers in a 2×2 matrix. Now we evaluate these three:

$$\begin{aligned} \vec{F} \times \vec{T} &= (-2 \times 6 - 4 \times -7)\hat{i} - (5 \times 6 - 4 \times 1)\hat{j} + (5 \times -7 - 1 \times -2)\hat{k} \\ &= 16\hat{i} - 26\hat{j} - 33\hat{k} \end{aligned}$$

So, finally this is our vector, and if we want to find the magnitude of it we can simply use equation (2), for any number of dimensions:

$$\begin{aligned} |\vec{F} \times \vec{T}| &= \sqrt{16^2 + (-26)^2 + (-33)^2} \\ &= 44.96 \end{aligned}$$

■

One another aspect in vector product that we need to determine the direction of the produced vector. We knew that the produced vector will be orthogonal to the plane of the two vectors, but we have two possibilities: the produced vector is pointing above the plane or under the plane. By convention, the right-hand rule is used to deduce the direction of the produced vector as shown in Figure 3.11. In calculating $\mathbf{a} \times \mathbf{b}$, we use the right hand, curl fingers starting at a and in the direction of b. The direction that the thumb points is the direction of $\mathbf{a} \times \mathbf{b}$. Next, $\mathbf{b} \times \mathbf{a}$: take the right hand, curl the fingers from b to a. Notice that the thumb points downwards, in the opposite direction of $\mathbf{a} \times \mathbf{b}$.

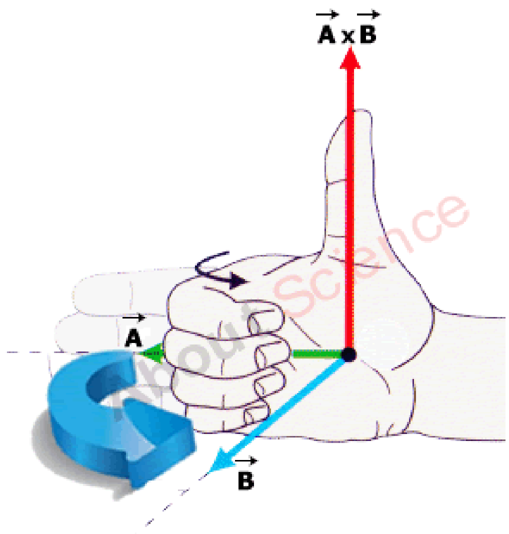


Figure 3.11: Right hand rule

Remark 3.4.

1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
2. The formula $|\vec{A} \times \vec{B}| = \|A\|\|B\| \sin\theta$ as shown gives only the magnitude of the vector. The vector itself is given by:

$$\vec{A} \times \vec{B} = \|A\|\|B\| \sin\theta \hat{n}$$

, where $\hat{n}$ is the unit vector of the produced vector.

3.3.4 The Scalar Triple Product

Here, we're going to put both the dot product and the cross product to use. Any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ in three dimensions determine a parallelepiped, a three-dimensional parallelogram-like box, as shown in Figure 3.12.

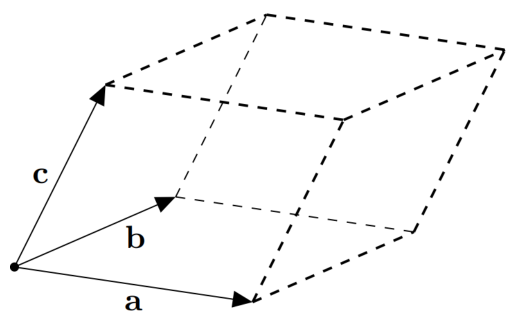


Figure 3.12: A parallelepiped

Let's calculate its volume. We know that:

$$V = (\text{Area of base})(\text{Vertical Height}) \tag{3.15}$$

Note that the base is the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. We can calculate its area by finding the magnitude of the cross product of $\mathbf{a}$ and $\mathbf{b}$ ($\|\mathbf{a} \times \mathbf{b}\|$). So equation 3.15 becomes:

$$V = \|\mathbf{a} \times \mathbf{b}\| \, (\text{Vertical height})$$

Next, we need to calculate the vertical height of the parallelepiped. In other words, we need to calculate the component of **c** in the direction perpendicular to the base. Explicitly, we want the component of **c** in the direction of $\mathbf{a} \times \mathbf{b}$. In particular, we want the length of the projection of **c** onto $\mathbf{a} \times \mathbf{b}$ as shown in Figure 3.13.

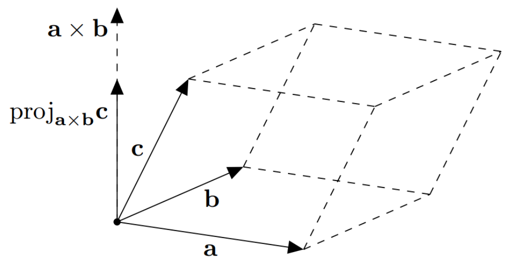


Figure 3.13: The height of a parallelepiped using the projection of vector **c**

From equation 3.12, we can say that the projection of **c** onto $\mathbf{a} \times \mathbf{b}$ is:

$$\text{Proj}_{a\times b}c = \frac{(a \times b) \cdot c}{\|a \times b\|}\hat{n}$$

Since we need the magnitude only of the projection, so:

$$\begin{aligned}\left\|\text{Proj}_{a\times b}c\right\| &= \left\|\frac{(a \times b) \cdot c}{\|a \times b\|}\hat{n}\right\| \\ &= \frac{|(a \times b) \cdot c|}{\|a \times b\|}|\hat{n}| \\ &= \frac{|(a \times b) \cdot c|}{\|a \times b\|}\end{aligned}$$

Then, the volume of the parallelepiped is:

$$V = \|a \times b\|\left\|\text{Proj}_{a\times b}c\right\| = \cancel{\|a \times b\|}\frac{|(a \times b) \cdot c|}{\cancel{\|a \times b\|}} = |(a \times b) \cdot c| \tag{3.16}$$

Equation 3.16 shows the formula for calculating the volume of any parallelepiped spanned by any three vectors **a**, **b**, and **c**. We could define the base of the solid differently, so the previous formula can be:

$$V = |(a \times b) \cdot c| = |(a \times c) \cdot b| = |(b \times c) \cdot a|$$

because we are taking the absolute value of the scalar triple product, the order doesn't matter.

Chapter 4

Motion in 2 and 3 Dimensions

Contents

4.1 The Position, Velocity, and Acceleration Vectors

In one dimension, a single numerical value describes a particle's position, but, in two dimensions, we indicate its position by its position vector $\vec{r}$ drawn from the origin of some coordinate system to the location of the particle in the $x-y$ or $x-y-z$ in three dimensions plane. If a particle is moving starting at some position A, until it reaches B at a time interval Δt. The position vector of the particle changes from A to B at Δt. The difference between the initial and final position of a particle is the **displacement vector** $\Delta\vec{r}$ as shown in Figure 4.1.

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i \tag{4.1}$$

If we want to quantify this change in position, it is useful to use the concept of **rate of change of position**. That is, dividing the **displacement** made by a particle over the time interval which it covers this displacement in. This quantity is identified to be **the average velocity** $\vec{v}_{avg}$

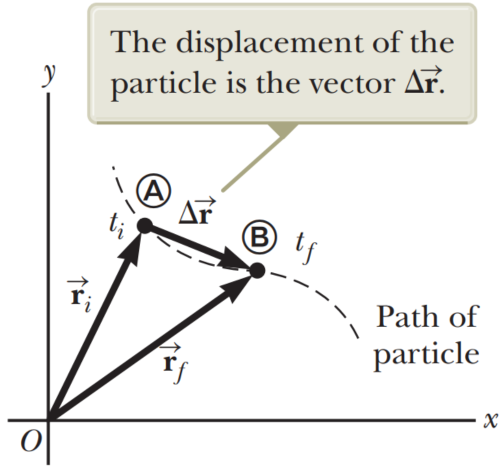


Figure 4.1: The particle's path from its initial to final position

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} \tag{4.2}$$

Multiplying or dividing a vector by a scalar such as Δt changes only the magnitude of the vector, not its direction. The average velocity is *independent* of the path taken, it depends on displacement which is the difference between the final and initial positions.

If we take the limit of the average velocity when Δt approaches zero, so we have the **instantaneous velocity** $\vec{v}$:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \tag{4.3}$$

Consequently, the change in velocity vector $\vec{v}$ over a defined time interval Δt is the so-called **average acceleration** $\vec{a}_{avg}$:

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} \tag{4.4}$$

Again, since the acceleration is the difference between two vector quantities, the average acceleration is a vector quantity. If we diminished the time interval t approach zero, we get the **instantaneous acceleration** $\vec{a}$:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{4.5}$$

4.2 Two-Dimensional Motion with Constant Acceleration

First of all, a generalization can be made which is that motion in two dimensions can be modeled as two independent motions in each of the two perpendicular directions associated with the x and y axes. That is, any influence in the y direction does not affect the motion in the x direction and vice versa.

Since we know that $\vec{v}_f = \vec{v}_i + \vec{a}t$. From vector addition, we can do the addition of two vectors if we add their components of the same dimension together:

$$\vec{v}_f = (\vec{v}_{xi} + \vec{a}_x t)\hat{i} + (\vec{v}_{yi} + \vec{a}_y t)\hat{j} \tag{4.6}$$

The same concept applies to all the kinematic equations when they are represented as vectors. Let's have an example.

Example 4.1

A particle moves in the $x-y$ plane, starting from the origin at $t = 0$ with an initial velocity having an x component of $20m \cdot s^{-1}$ and a y component of $215m \cdot s^{-1}$. The particle experiences an acceleration in the x direction, given by $a_x = 4.0m \cdot s^{-2}$. (A) Determine the total velocity vector at any later time. (B) Calculate the velocity and speed of the particle at $t = 5.0s$ and the angle the velocity vector makes with the x axis.

Solution. (A) Since, $\vec{v}_f = (\vec{v}_{xi} + \vec{a}_x t)\hat{i} + (\vec{v}_{yi} + \vec{a}_y t)\hat{j}$. Therefore,

$$\vec{v}_f = (20 + 4t)\hat{i} - 15\hat{j}$$

(B) At $t = 5s$, the velocity will be:

$$\vec{v}_f = (20 + 4(5))\hat{i} - 15\hat{j} = 40\hat{i} - 15\hat{j}$$

To find the angle θ between the velocity vector at $t = 5s$ and the x axis, we do:

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15}{40}\right) = -21^{\circ}$$

■

4.3 Projectile motion

Projectile motion is a special case of two-dimensional motion. A particle moving in a vertical plane with an initial velocity and experiencing a free-fall (downward) constant acceleration, displays projectile motion. We find that the path of a projectile, which we call its trajectory, is always a **parabola**. Of course, to describe motion we must deal with velocity and acceleration, as well as with displacement. We must find their components along the $x-$ and $y-$ axes, too. We will assume all forces except gravity (such as air resistance and friction, for example) are negligible. The components of acceleration are then very simple: $a_y = -g = -9.80m \cdot s^{-2}$. Because gravity is vertical, $a_x = 0$. Both accelerations are constant, so the kinematic equations can be used.

Remark 4.1. This definition assumes that the upwards direction is defined as the positive direction. If you arrange the coordinate system instead such that the downwards direction is positive, then acceleration due to gravity takes a positive value.

Remark 4.2. According to these assumptions, some procedures can be taken to approach projectile motion problems:

1. Resolve or break the motion into horizontal and vertical components along the x- and y-axes. These axes are perpendicular, so $v_x = v\cos\theta$ and $v_y = v\sin\theta$.
2. Treat the motion as two independent one-dimensional motions, one horizontal and the other vertical. Use SUVAT equations to analyze these motions.
3. Solve for the unknowns in the two separate motions, one horizontal and one vertical. Note that the only common variable between the motions is time t.
4. Recombine the two motions to find the total displacement (s) and velocity (v) by vector addition rules.

Example 4.2

In a football match, the ball went out of the field of play. So, the goalkeeper had to play a goal kick. The ball is kicked until it fell exactly at the end of the court on the other side, and it went out again. Knowing that the initial velocity was $25m \cdot s^{-1}$, and it made an angle of $45°$ with the horizontal, determine the time of flight, total length of court covered, and the peak height of the ball. (Assume that air resistance is negligible)

Solution. We can divide the problem into two independent motions, horizontal and vertical motion. Since horizontal motion has no acceleration, one simple formula can be used, which is:

$$s_x = v_x t \tag{4.7}$$

We can solve for v_x as $v_x = V\cos\theta$. So,

$$v_x = 25m \cdot s^{-1} \cos 45 = \frac{25\sqrt{2}}{2} m \cdot s^{-1}$$

Then, we must have the time of flight to determine the horizontal distance covered (the length of the court). So, we can go into vertical motion to solve for time. The most suitable kinematic equation is:

$$\mathit{\Delta} s = v_i t + \frac{1}{2} a t^2$$

Since the ball is projected from the ground and falls again at the end of the flight, so $\mathit{\Delta} s = 0$, $v_{iy} = V\sin\theta$, and $a = -g$. Then, we can solve for time:

$$\begin{aligned} 25\sin 45t - \frac{1}{2}9.8t^2 &= 0 \\ t\left(25\sin 45 - \frac{1}{2}9.8t\right) &= 0 \end{aligned}$$

We have two solutions for the above equation: $t = 0$, and this is refused. The other solution is the real time taken:

$$\begin{aligned} 25\sin 45 - \frac{1}{2}9.8t = 0 \\ t = \frac{25\sin 45}{0.5 \times 9.8} \approx 3.61 \text{ s} \end{aligned}$$

Substituting in the kinematic equation of displacement, we get the length of the court:

$$s_x = \frac{25\sqrt{2}}{2} m \cdot s^{-1} \times 3.61s \approx 63.78 \text{ m}$$

For the height of the peak of the ball, we know that the y-component velocity = 0, so we can make use of this SUVAT equation:

$$\begin{aligned} v_f^2 - v_i^2 &= 2a\Delta s \\ 0^2 - (25\sin 45)^2 &= 2 \times -9.8 \times \Delta s \\ \Delta s &= \frac{(25\sin 45)^2}{2 \times 9.8} \approx 15.94 \text{ m} \end{aligned}$$

■

4.3.1 A special Projectile Motion Case

One interesting type of projectile motion is the projectile that is launched from the origin at $t_i = 0$ with a positive v_{yi} component, as shown in Figure 4.2, and returns to the same horizontal level. Two points in this motion are especially interesting to analyze: the peak point $\boldsymbol{A}$, which has Cartesian coordinates $(R/2,\, h)$, and the point $\boldsymbol{B}$, which has coordinates $(R,\, 0)$.

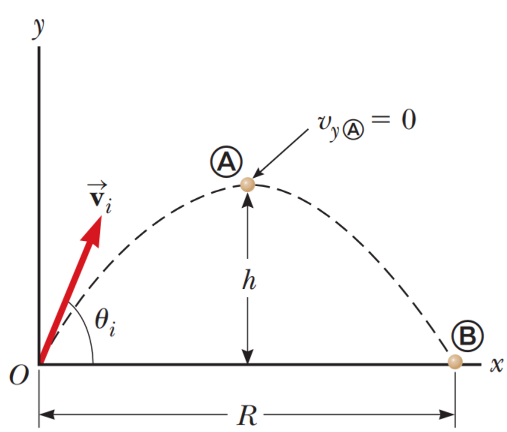


Figure 4.2: A projectile launched over a flat surface from the origin

When we need to calculate the maximum height, h, of the trajectory (Point A), we make use of this kinematic equation:

$$v_{fy}^2 - v_{iy}^2 = -2gh$$

The y-component of the velocity at the peak point is zero because the tangent of the path of the trajectory (y-component velocity vector) is decreasing until it becomes parallel to x-axis, and thus zero at exactly the maxima of the curve.

The initial velocity is $v_{iy} = v \sin\theta$. So:

$$0^2 - (v\sin\theta)^2 = -2gh$$

$$h = \frac{v^2 \sin\theta^2}{2g} \tag{4.8}$$

To calculate the horizontal range R until point B, we use the formula for the horizontal motion:

$$s_x = v_x t \tag{4.9}$$

But we need the total time of the flight. So, we can get the time using vertical motion by equation:

$$\mathit{\Delta} s = v_{iy} t - \frac{1}{2} g t^2$$

Since the displacement is zero because the trajectory returns to the same horizontal level. And $v_{iy} = v \sin\theta$, then:

$$0 = t\left(v\sin\theta - \frac{1}{2} g t\right)$$

$$t = \frac{2v\sin\theta}{g} \tag{4.10}$$

Substituting equation 4.10 for time in equation 4.9 for horizontal range:

$$s_x = v\cos\theta \times \frac{2v\sin\theta}{g} = \frac{v^2(2\sin\theta\cos\theta)}{g}$$

Using the double angle identity for the sine function:

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{4.11}$$

We get:

$$R = \frac{v^2 \sin(2\theta)}{g} \tag{4.12}$$

So, we come up with two formulas, equation 4.8 and 4.11 for the maximum height and the range of the path respectively, **BUT** in the special case of full trajectory path returning to the same horizontal level again.

Claim 4.1 — Let a trajectory that is thrown and stop at the same horizontal level, so the maximum height of the flight h and the range of the motion R given by:

$$R = \frac{v^2 \sin(2\theta)}{g}$$

$$h = \frac{v^2 \sin\theta^2}{2g}$$

Remark 4.3. From 4.11, we conclude that at angle theta, there will be two angles give exactly the same horizontal range. These are the two complementary angles as shown in Figure 4.3.

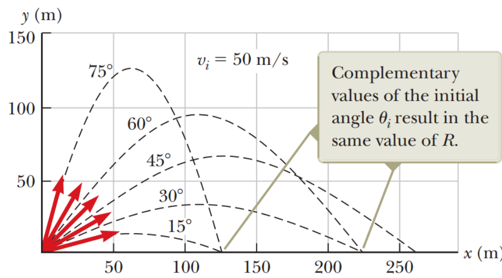


Figure 4.3: Complementary trajectory angles give the same range

4.4 Relative Velocity and Relative Acceleration

Relative motion, as the name implies, is the motion of an object relative to another observer. The observer is a spare term for the frame of reference. The concept applies to relative positions, velocities, and accelerations.

As shown in Figure 4.4, Consider a particle P and reference frames S and S'. The position of the origin of S' as measured in S is $\vec{r}_{S'S}$, the position of P as measured in S' is $\vec{r}_{pS'}$, and the position of P as measured in S is $\vec{r}_{ps}$.

From Figure 4.4, we see that:

$$\vec{r}_{ps} = \vec{r}_{ps'} + \vec{r}_{s's} \tag{4.13}$$

Equation 4.13 is the formula for **Relative positions**. Since velocity is the derivative of position, we can deduce that:

$$\vec{v}_{ps} = \vec{v}_{ps'} + \vec{v}_{s's} \tag{4.14}$$

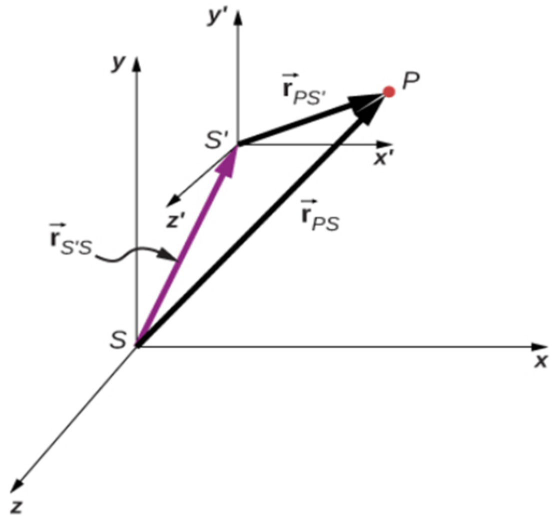


Figure 4.4: Representation of position of a particle (P)

This formula can apply to any number of reference frames. If we extend the formula for relative acceleration, since acceleration is the derivative of velocity, we have:

$$\vec{a}_{ps} = \vec{a}_{ps'} + \vec{a}_{s's} \tag{4.15}$$

If the velocity of $S^{'}$ is constant relative to S, then $\vec{a_{s's}} = 0$. So, $\vec{a_{ps}} = \vec{a_{ps'}}$. That is, the acceleration of the particle measured by an observer in one frame of reference is the same as that measured by any other observer moving with constant velocity relative to the first frame.

Equation 4.14 for relative velocities applies to motion in one or more dimensions. In 1-D, velocities are simply added or subtracted based on their signs, as they are either in exactly the same direction or in opposite directions. In $2-D$, we make use of vector addition principles to evaluate the relative velocity. Let's look at an example.

Example 4.3

A truck is traveling south at a speed of 70 $km \cdot h^{-1}$ toward an intersection. A car is traveling east toward the intersection at a speed of 80 $km \cdot h^{-1}$. What is the velocity of the car relative to the truck?

Solution. We have a specific formula for finding relative velocity. Assuming the subscripts C, T, and E stand for Car, Truck, and Earth respectively, we can write:

$$\vec{v}_{CT} = \vec{v}_{CE} + \vec{v}_{ET}$$

Note that $\vec{v_{ET}} = -\vec{v_{TE}}$, so if we reverse the direction of the truck as shown in Figure 4.5 below, we can represent the vector addition for the velocities. We can now solve for the velocity of the car with respect to the truck:

$$|\vec{v_{CT}}| = \sqrt{(80\ km \cdot h^{-1})^2 + (70\ km \cdot h^{-1})^2} \approx 106.3\ km \cdot h^{-1}.$$

Now that we have found the magnitude of the velocity, we need to determine its direction to represent the velocity vector.

We can find the angle by:

$$\theta = \tan^{-1}\left(\frac{70}{80}\right) = 41.2^{\circ} \text{ north of east}$$

■

Chapter 5

The Laws of Motion

Contents

5.1 Forces

Published in the 17th century, Sir Isaac Newton's *Principia Mathematica* explained three laws of motion, introducing the concept of *force*. In our day-to-day lives, we perceive forces as the pulling or pushing that causes motion, but that's not an accurate definition of forces. While forces are capable of causing motion, they do not always do so (when balanced by other forces). For example, you can apply a force on a wall (by pushing it) and not cause it to move.

Definition 5.1. A force is a *vector* quantity that changes an object's state of motion (moving or resting), direction of motion, and/or velocity (i.e., create acceleration).

Forces have can be classified into two types: *contact forces* and *field forces*. Contact forces are the "intuitive" type; they involve direct interactions (i.e., contact) between two objects. Examples for contact forces are the (1) applied, (2) perpendicular-to-surface normal (between a resting book and a table), (3) parallel-to-surface friction (between a sliding book and a table), (4) and string-transmitted tension forces.

Field forces, also known as *long-range forces*, do not involve physical contact; they act through space (i.e., do not need a medium). Examples for field forces are the (1) gravitational, (2) electric, and (3) magnetic forces.

However, this contact-or-field classification is not as accurate as it may appear. On an atomic scale, "contact" forces are just numerous "field" electric forces that take place between the atoms' charged particles. A more accurate classification of forces goes as follows: the only four fundamental forces are (1) gravitational forces (between masses), (2) electromagnetic forces (between charged particles), (3) strong nuclear forces (that hold *quarks* and/or *nucleons* together), and (4) weak nuclear forces (that cause particles' decay into other particles).

5.2 Newton's First Law of Motion

Newton's first law of motion, also known as the *law of inertia*, can be understood after understanding *reference frames* and their types. A reference frame is a (most likely Cartesian) coordinate system with reference points (e.g., the Earth's surface) that define that coordinate system's origin, orientation, and scale. There are two types of reference frames: *inertial* and *non-inertial*.

An inertial reference frame is a reference frame that is not accelerating. However, a vector quantity such as acceleration must be measured with respect to other reference frames, so how do we define an inertial reference frame? We can assume that a particular reference frame is inertial, which will allow us to define other inertial reference frames.

Definition 5.2. An inertial reference frame is a reference frame that is not accelerating (at rest or moves with constant velocity) with respect to an assumed-as-inertial reference frame.

Examples for assumed-as-inertial reference frames are the "fixed" stars and the Earth (which is actually non-inertial, as it orbits the sun). Conversely, a non-inertial reference frame is defined as follows:

Definition 5.3. A non-inertial reference frame is a reference frame that is accelerating (not at rest and not moving with constant velocity) with respect to an assumed-as-inertial reference frame.

Finally, we can understand why Newton's first law of motion is often called the *law of inertia*:

> **Claim 5.1 — Newton's First Law of Motion**: In the absence of external forces and when viewed from an inertial reference frame, an object does not accelerate (i.e., resting objects remain resting, and moving objects remain moving with the same velocity).

When viewed from a non-inertial reference frame, however, *fictitious forces* (also known as *pseudo forces*) appear, which are apparent "forces" (not real forces) that are due observing in a non-inertial (accelerating) reference frame. Normally, fictitious forces' directions appear to be opposite to the direction of the acceleration of the non-inertial reference frame. For example, imagine yourself driving a car that moves with a constant velocity in a straight line. If you suddenly took a sharp left turn (i.e., the car accelerated to the left), you would "slide" to the right. You didn't actually "move" (at least with respect to an inertial reference frame); no actual forces "moved" you to the right. What happened is that you felt being pushed to the right because your non-inertial reference frame (the accelerating car) moved to the left with respect to a non-accelerating inertial frame of reference.

Definition 5.4. Fictitious forces are apparent (not real) forces that are perceived due to observing with respect to an accelerating non-inertial reference frame. Their directions is oftentimes the opposite of the acceleration of the non-inertial reference frame.

5.3 Newton's Second Law of Motion

Forces can be added *vectorially* (since forces are vectors). The sum of forces acting on an object are defined as the *net force* (also called *total force* or *resultant force*). When the net force equals zero, the forces' effects cancel out and are, therefore, said to be balanced (i.e., do not change the object's state of motion). When you apply a force on a wall (by pushing it), the wall does not move because your applied force and the normal force cancel out (i.e., the wall's net force is zero).

> **Remark 5.1. The net force** $\vec{\mathrm{F}}_{\text{net}}$ acting on an object is given by
>
> $$\vec{\mathrm{F}}_{\text{net}} = \sum_i \vec{\mathrm{F}}_i \tag{5.1}$$
>
> where $\vec{\mathrm{F}}_i$ is the ith force acting on the object, and $\sum_i \vec{\mathrm{F}}_i$ represents the sum of all the forces acting on the object.

You might have already guessed that the net force (rather than the ith force) is the quantity that governs the object's motion changes. Perhaps you have guessed that it is related to the object's acceleration. Newton's second law of motion states that the net force acting on an object is proportional to the object's mass m and acceleration a.

> **Claim 5.2 — Newton's Second Law of Motion**: The net force $\vec{\mathrm{F}}_{\text{net}}$ acting on an object is given by
>
> $$\vec{\mathrm{F}}_{\text{net}} = m\vec{a} \tag{5.2}$$
>
> where m is the object's mass, and $\vec{a}$ is the object's acceleration.

From this definition, we can infer three things:

1. Mass is the property that resists changes in velocity. The greater the mass an object has, the less it will accelerate (given that the force is constant). This explains why, for example, a car accelerates less than a tennis ball when the same force is applied on the two objects.

2. The greater the net force, the greater the acceleration, which is intuitive. For example, the more you push a car (increase the net force), the more it will accelerate.

3. Forces' SI measuring unit, the newton (N), is equivalent to $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$.

Newton's second law is a powerful tool that is going to be useful for the rest of your Newtonian mechanics learning journey. Take this light exercise as an example of the usefulness of Newton's second law.

Example 5.1

Mina and Mustafa fight over which direction a 1,000 kg car should go. Mina pushes the car with 100 N, and Mustafa pushes the car in the opposite direction with a force $\vec{\mathrm{F}}_{\mathrm{M}}$. If you know that the car is accelerating with a magnitude of $0.4\ \mathrm{m} \cdot \mathrm{s}^{-2}$ towards Mina, find $\vec{\mathrm{F}}_{\mathrm{M}}$?

Solution. First, note that $a = -0.4$, because it's opposite to Mina's positive force (signs imply directions). The sum of Mina and Mustafa's forces must equal the net force $\vec{\mathrm{F}}_{\mathrm{net}} = m\vec{a}$:

$$100 + \vec{\mathrm{F}}_{\mathrm{M}} = 1000 \times (-0.4) \implies \vec{\mathrm{F}}_{\mathrm{M}} = -(1000 \times 0.4 + 100) = -500\ \mathrm{N}$$

Mustafa pushes with a force of 500 N towards Mina, which is opposite to the direction Mina's force. ■

In our everyday language, we often interpret an object's "weight" as its mass. However, there is a distinction between the two. An object's mass is an *intrinsic* (i.e., inherent) property measured in kg. Weight, an *extrinsic* (i.e., not inherent) property measured in N, is the *magnitude* of the gravitational force $\vec{\mathrm{F}}_g = m\vec{g}$ (given that no other forces are acting), where m is the mass of the object, and $\vec{g}$ is the gravitational acceleration.

5.4 Newton's Third Law of Motion

Ever wondered why punching something hurts? When you push something, that things *pushes you* back. Newton's third law of motion, also known as the action-reaction law, states that every *acting force* has a *reacting force* that is equal in magnitude and opposite in direction. Mathematically, Newton's third law is expressed as the following:

Claim 5.3 — Newton's Third Law of Motion: If two objects named 1 and 2 interact, the force $\vec{\mathrm{F}}_{1\text{ on }2}$ exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force $\vec{\mathrm{F}}_{2\text{ on }1}$ exerted by object 2 on object 1:

$$\vec{\mathrm{F}}_{1\text{ on }2} = -\vec{\mathrm{F}}_{2\text{ on }1} \tag{5.3}$$

where signs imply directions.

An example for this action-reaction phenomenon is gravity. As the Earth pulls you (gravitationally), you pull the Earth as well. You might wonder why doesn't the Earth "get pulled" as much as you do. That is because the Earth's mass ($= 6 \times 10^{24}$ kg) is much greater than your mass (from 2.3: the greater the mass, the less the acceleration).

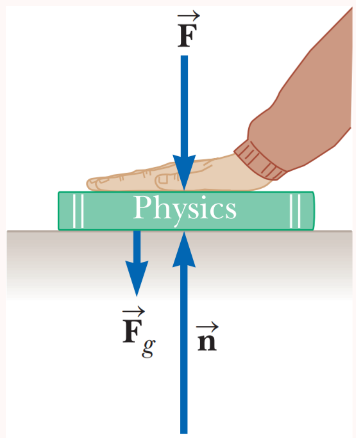
A popular misconception is that the *normal force* is the reaction of the gravitational force. This is false. A normal force does not occur while something is falling, for example. If the normal force was the reaction of the gravitational force, then this non-existent normal force while falling would violate Newton's third law. When an object gets pulled to the Earth due to gravity, that object pulls Earth; that object's gravitational pull (not the normal force!) is the reaction to the Earth's gravitational pull.

A normal force is the reaction to the contact force that an object does to a surface. For example, imagine sitting on a chair. You won't find yourself going through the chair; you'll be balanced (at least vertically). That is because the chair's *reacting* normal force is equal in magnitude and opposite in direction to your *acting* (pushing) contact force upon it.

To not get confused when trying to find out the reacting force to a particular acting force, remember this rule of thumb: *An acting contact force has a reacting contact force, and an acting field force has a reacting field force.* See 2.1 to read about contact and field forces.

Example 5.2

You push downwards with force of $\vec{F} = 50$ N on a resting physics book that weighs $\vec{F}_g = 3$ N. What is magnitude of the normal force $\vec{n}$? What is the *acting* force of that *reacting* normal force?



Solution. The book being "resting" implies that the net force is equal to zero:

$$\sum_i \vec{F}_i = 0 \implies \vec{F}_g + \vec{F} + \vec{n} = 0 \implies \vec{n} = -53 \text{ N}$$

The negative sign implies that $\vec{n}$'s direction is upwards, opposite to the pushing. The normal force, in this case, is the reaction of the book-on-surface overall pushing, which is $\vec{F}_g + \vec{F} = 53$ N. Note the opposite signs of $\vec{n}$ and $\vec{F}_g + \vec{F}$. ■

Chapter 6

Circular Motion and Other Applications of Newton's Laws

Contents

6.1 Dynamics of uniform circular motion

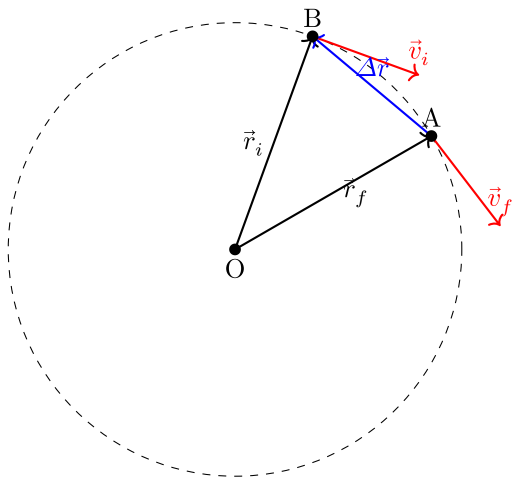
in the previous chapters you were introduced to the motion in one direction. imagine a car taking the ring road assuming it is perfectly circular. that is what we call a circular motion. if the car has a constant velocity v the car is said to be in uniform circular motion. despite that the car has a constant velocity, it still has an acceleration. recall the definition

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

since the car is moving in a circular shape the direction of the velocity changes resulting in a change of velocity and causing an acceleration. let us try to find the acceleration. the figure below shows the motion of of the car in the circular motion when you make the velocity vector triangle. it would be similar to that of the radii vector triangle. thus, a similarity can be made.

$$\frac{\vec{v}_i}{\vec{r}_i} = \frac{\vec{v}_f}{\vec{r}_f} = \frac{\Delta\vec{v}}{\Delta\vec{r}}$$

Figure 6.1: Circuar motion up-close



$$|\vec{a}| = \frac{v\Delta r}{r\Delta t =} = \frac{v^2}{r}$$

the period of the motion T which is the amounts of seconds per one revolution can be found by:

$$T = \frac{2\pi r}{v}$$

the angular speed ω:

$$\omega = \frac{2\pi}{T}$$

to find the relation between ω and v:

$$\omega = 2\pi \times \frac{1}{T} = 2\pi \times \frac{v}{2\pi r} = \frac{v}{r}$$

thus:

$$v = r\omega$$

the equations demonstrates that the translational velocity becomes large as the radius and the angular velocity become larger

the centripetal acceleration can be further expressed by

$$a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

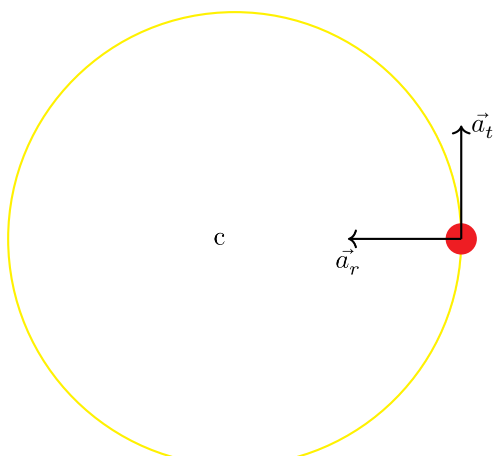
6.2 Non-uniform circular motion

In the previous section, we introduced the concept of uniform circular motion where the object moves in a circular path while the magnitude of its velocity is constant. In this section, we will discuss the concept of non-uniform circular motion where the magnitude of velocity is not constant; it changes due to tangential acceleration produced by external force like gravity or friction. Note that in real life situations, the non-uniform circular movement is almost always the case.

> **Remark 6.1.** Because we are not dealing with an ideal system in the real world, the object in the circular motion is always affected by some external force.

Consider an object moving in a circular path with tangential acceleration of a_t and centripetal acceleration of a_c as in Figure (1).

Figure 6.2: A body moving in a circular path with centripetal and tangential acceleration



Suppose that at time $t = 0$ the velocity of the body was v_i and a tangential acceleration of a_t. The centripetal force at $t = 0$ is simply:

$$a_c = \frac{v_i^2}{r}$$

However, due to a_t this centripetal acceleration isn't constant. As the velocity v_i changes with time, the centripetal acceleration changes. The total acceleration is simply:

$$\vec{a_T} = \vec{a}_c + \vec{a}_t \tag{6.1}$$

We can get the value of the total acceleration by getting the resultant of the two vectors:

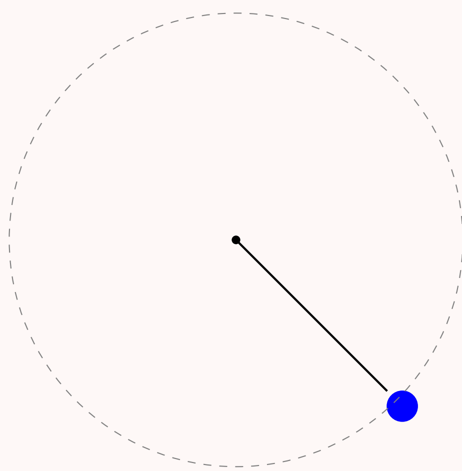
$$a_T = \sqrt{a_t^2 + a_c^2} \tag{6.2}$$

Example 6.1

A pendulum is moving around a circular path with a radius r and the mass of the pendulum bob is m. What is the tension force at the moment when the pendulum has moved by angel θ and its velocity is v at this moment? neglect the mass of the rod.

Solution This is a realistic example of non-uniform circular motion. Besides the centripetal force, the pendulum is affected by gravity. Figure 4.2 shows the pendulum in its initial position while making angle of 0. In its second positions

Figure 6.3: A body moving in a circular path with centripetal and tangential acceleration



6.3 Rotational dynamics

Claim 6.1 — The moment of inertia, often denoted as (I), is a property of an object that quantifies its resistance to rotational motion about a particular axis.

Inertia measures how the mass of an object is distributed in relation to that axis. The moment of inertia depends on both the mass of the object and how that mass is distributed with respect to an axis. Mathematically, the moment of inertia is described as the sum of the product of mass and the square distance from the axis.

$$I = \sum mr^2 \tag{6.3}$$

In more complex systems, we use calculus. Equation 21 can be used in an integral form. Assume that the mass becomes infinitesimally small dm and r is the distance between the axis and the small mass dm.

$$I = \int r^2 dm \tag{6.4}$$

Let's consider an object with moment of inertia I_{CM} about an axis passing through its center of mass (CM). Now, we want to find the moment of inertia I_{P} of the same object about a parallel axis P that is a distance d away from the center of mass.

The additional moment of inertia I_{P} can be calculated by considering each particle of mass m in the object. The perpendicular distance (r) between the particle and the parallel axis P is given by $r = d + r_{\mathrm{CM}}$, where r_{CM} is the perpendicular distance between the particle and the center of mass axis.

Using the integral definition of moment of inertia, we have:

$$I_{\mathrm{P}} = \int r^2 dm = \int (d + r_{CM})^2 dm$$

Expanding the square and grouping terms, we get:

$$I_{\mathrm{P}} = \int (d^2 + 2dr_{\mathrm{CM}} + r_{\mathrm{CM}}^2) dm$$

$$I_{\mathrm{P}} = d^2 \int dm + 2d \int r_{CM} dm + \int r_{\mathrm{CM}}^2 dm$$

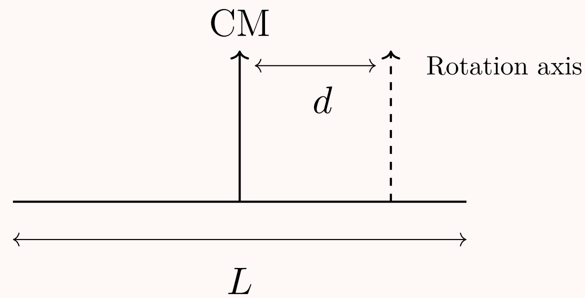
Since $\int r_{CM} dm = 0$ (by definition of center of mass), $\int dm = M$, and third term is the moment of inertia around the center of mass, we can simplify the expression:

$$I_{\mathrm{P}} = I_{CM} + Md^2 \tag{6.5}$$

This equation gives the moment of inertia of an object about an axis P parallel to the axis in the centre of mass.

Example 6.2

A thin uniform rod of mass M and length L rotates about an axis that is perpendicular to the rod. The axis passes through a point at a distance d from the rod's center (along its length).



Knowing that the rod's moment of inertia about its CM axis is $I_{\mathrm{CM}} = \frac{1}{12}ML^2$

Calculate the moment of inertia of the rod about the given axis.

Solution. Let the rod lie along the x-axis with its center at the origin. The axis of rotation is at $x = d$. Divide the rod into infinitesimal elements dm of length dx, where:

$$dm = \frac{M}{L}dx$$

The distance from the axis of rotation to a small element at position x is:

$$r = |x - d|$$

The moment of inertia is given by:

$$I = \int r^2 dm = \int_{-L/2}^{L/2} (x-d)^2 \frac{M}{L} dx$$

Expanding $(x-d)^2$, we have:

$$I = \frac{M}{L} \int_{-L/2}^{L/2} \left(x^2 - 2xd + d^2\right) dx$$

Split the integral:

$$I = \frac{M}{L} \left[\int_{-L/2}^{L/2} x^2 dx - 2d \int_{-L/2}^{L/2} x dx + d^2 \int_{-L/2}^{L/2} dx \right]$$

Compute each term:

$$\int_{-L/2}^{L/2} x^2 dx = \frac{1}{3}\left(\frac{L}{2}\right)^3 + \frac{1}{3}\left(\frac{L}{2}\right)^3 = \frac{1}{12}L^3,$$

$$\int_{-L/2}^{L/2} x dx = 0 \quad \text{(symmetry)},$$

$$\int_{-L/2}^{L/2} dx = L.$$

Substitute back:

$$I = \frac{M}{L}\left[\frac{1}{12}L^3 + 0 + d^2 L\right] = \frac{1}{12}ML^2 + Md^2$$

The moment of inertia about the center of mass is:

$$I_{\text{CM}} = \frac{1}{12}ML^2$$

The parallel axis theorem states:

$$I_{\text{P}} = I_{\text{CM}} + Md^2$$

Substituting I_{CM}:

$$I_{\text{P}} = \frac{1}{12}ML^2 + Md^2$$

This matches the result obtained via direct integration.

■

Chapter 7

Energy of a System

Contents

7.1 Systems

Before going in the details of work and Energy, we have to define what a system is.

Definition 7.1. System is a part of the universe that we focus our attention on with the ignoring of the other parts of the universe.

A system can be an one object that we focus on to calculate its speed. A system can be two or more objects colliding with each other that we focus on to measure their momentum. It can be an enclosed section in space like a room that we focus on to analyze its temperature. It can be an open section of space with no real boundaries, like a planet, that we focus on to depict gravitational effects. It can be the universe itself when dealing with cosmological bodies.

> **Remark 7.1.** The universe is the biggest system that can be modeled.

The boundaries of the system can be real like the walls of the room or imaginary like the boundaries between two colliding balls. Those boundaries sepa-

rate our system from the **environment** surrounding it. For example, we can represent our system as a standing man. The boundaries of the man's body is the system's boundaries. The environment in this case is everything that is not the man's body. when we say that a force is applied on the object by an external source, we can represent it as the environment is exerting force on the system's boundaries, which caused the system to feel this force.
There are many ways by which the environment can influence the system and vice-versa. We will discuss work as the first.

7.2 Work

To remove any upcoming misconception, the term "work" in physics has no relation with the one used in the daily life as job or career. To understand what "work" in physics mean, we shall illustrate the next situation.

Consider there is a book on a table and we are pushing that book with a force **F** to move it to a displacement of **d**. If we want to know how effective this force is in moving the book, we won't consider only the force but also how it influenced the object (how effectively it changed its position). If we apply the same same force on the book in two different cases, one the book moves a distance d and in the other the book moves 2d, which is more effective?. Obviously, the case when the object moves a larger distance. Similarly, moving the book a displacement d with a force of 2F makes greater influence on the system that when applying a force F. Also, if there is no force applied or the position of the book is unchanged, the effect of the environment on the system is zero. From this, we can conclude that the influence on the system is proportional to the force and to the displacement.

Consider a case where the force acting on the book is making an angle θ with the direction of the displacement. The force affecting the book in the direction of motion is the component of force in the direction of motion—the component parallel to the direction of motion. This component equals $F\cos(\theta)$.

Corollary 7.1 (Work)

Work done on a system (**W**) by a constant force from an outer environment is equal to the product of force (**F**), displacement by the object (**d**), and the angle between the displacement and force vector ($\cos(\theta)$).

$$W = Fd\cos(\theta) \tag{7.1}$$

To further illustrate the difference between work in physics and in common language, imagine yourself holding a dumbbell in the same position for a minute. You will feel tired and feel like you have done a huge work, but in our definition for work, you did zero work as the dumbbell had 0 displacement. This is because *work* in physics doesn't represent how hard or tiring the action but how much does it influence the system.

Remark 7.2. The work done by the gravitational force is zero because the force is perpendicular to the displacement, making $\cos(\theta) = 0$

The SI unit of the work is the units of the force $\times$ the units of the distance, which is Newton $\bullet$ meter (Kg $\bullet$ m^2/sec^2). As the work is frequently used in physics, it has its own unit which is **Joule (J)**.

Work can be thought of as money. It can be transferred from environment to a system, from a system to the environment, or from a system to another system. When consider Work done on a system, a positive sign indicates the transfer of energy to the system. A negative sign, similarly, indicated the transfer of energy from the the system. Energy is not yet explained but we can define it as a conserved physical quantity. In contrast, Work done by the system will have the opposite effects for positive and negative signs.

Example 7.1

A man is pulling a trolley with a constant force to the right of 10N, making an angle of 60° with the horizontal. Calculate the work done on the trolley when the trolley is displaced by 5 meters to the right. **Solution**: The angle between the force and the horizontal is the same angle between the force and the displacement as the displacement is on the horizontal. So,

$$W = Fd\cos(\theta)$$

$$= 10 \times 5 \times \cos(60) = 25J$$

7.3 Work Done by Variable Forces

We have discussed work previously as the product of the applied force and the displacement of the moving object. However, this is not always the case in many situations. Most of the time the object is displaced by a varying force which varies with its position. In this section, we will explain how to evaluate the work done by this type of forces.

Consider an object moving under some varying force $F(x)$ that changes with the position x. Since we can't compute the work done directly as the dot product between the force and the displacement vectors, we may divide the displacement of the object into small sections and sum up the work done in each section to get the work done. Let's take the work done on the x-axis, for example:

$$W = \sum_{x_i}^{x_f} F(x)\Delta x$$

If we increase the number of these section indefinitely to infinity decreasing the displacement of each section approaching zero, we would get a Riemann sum which is then converted into an integral:

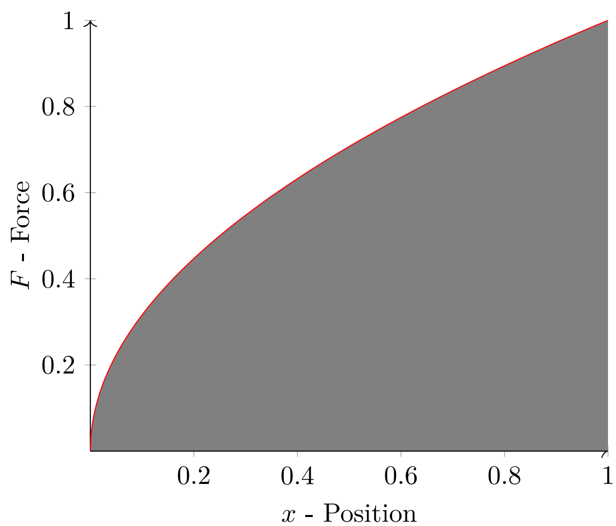
$$W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F(x)\Delta x = \int_{x_i}^{x_f} F(x)dx \tag{7.2}$$

A general equation for calculating the work done by some varying force $\sum F$ is:

$$W = \int \sum \vec{F} \cdot d\vec{r} \tag{7.3}$$

According to the previous equation, work done is actually the area under curve of the force as a function of the position as shown in Figure

Figure 7.1: The work done due to a varying force is the area under the curve of the force versus position graph



Example 7.2

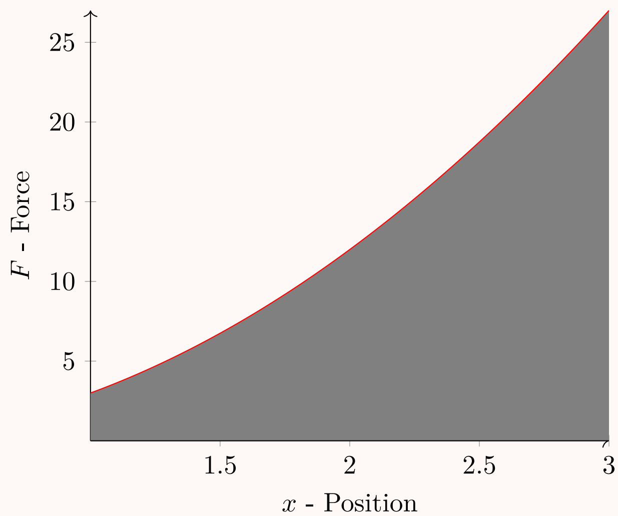
Consider an object acted on by a varying force that changes with position x according to the equation $F(x) = 6x^2$. If the force vector makes and angle $\theta = 60°$ with the x-axis, what is the work done by the x-component of the force to move the object from $x = 1$ to $x = 3$?

Solution

This is a simple problem in which we use **Equation 8.1** to solve:

$$W = \int_1^3 6x^2 \cos(60°) dx = \int_1^3 3x^2 dx = \left[x^3\right]_{x=1}^{x=3} = 26J$$

Figure 7.2: The work done due to a varying force is the area under the curve of the force versus position graph



7.4 Work-Energy theorem and Kinetic Energy

We have previously discussed how the energy transfer to the system, but we didn't describe its effect on the system nor how the system stores energy. One intuitional effect is the rise in the speed of the system when giving it energy. We will start by the first type of energy storage related to this example: **kinetic energy**.

To relate kinetic energy to work, consider a situation where a block of mass m is affected by a net external force of $\sum \mathbf{F}$ to the right. That implies that the block moves with acceleration $\mathbf{a}$, according to Newton's second law. If the block moves to the right through a displacement $r = (x_f - x_i)$- and has a velocity v_i at x_i and a velocity v_f at x_f, the net work done is given by equation 7.2:

$$W = \int \sum \vec{F} \cdot d\vec{r}$$

$$W = \int_{x_i}^{x_f} ma\, dr = \int_{x_i}^{x_f} m\frac{dv}{dt}\, dr = \int_{x_i}^{x_f} m\frac{dv}{dr}\frac{dr}{dt}\, dr = \int_{v_i}^{v_f} mv\, dv$$

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \tag{7.4}$$

Equation 7.4 is a general case that tells us that the change of work done on an object with mass m is equal to the difference of $\frac{1}{2}mv^2$, a quantity that is called **kinetic energy**.

$$KE = \frac{1}{2}mv^2 \tag{7.5}$$

Equation 7.5 changes Equation 7.4 to:

$$W_{external} = KE_{final} - KE_{initial} = \Delta KE \tag{7.6}$$

Corollary 7.2 (Work-Kinetic energy theorem)
The work-Kinetic energy theorem states that if work is done on a system and the system only changed part is its speed, the change in kinetic energy equals that work done, expressed in Equation 7.6.

The work-energy theorem indicates that when work done on the system is positive, the kinetic energy is increasing. In contrast, when it is negative, the kinetic energy is decreasing.

Work-energy theorem is valid for not only translational motion but also for rotational motion that will be discussed in later chapters. As stated earlier, work is a mechanism by which energy transfer. Equation 7.6 is the mathematical statement of the concept for one case, kinetic energy. In later sections, we are going to discuss other energy types a system can store as a result of work done on the system.

7.5 Potential Energy

Let's imagine a situation where a book is held above the earth and affected by its gravitational force. We will consider the initial position of the book as y_i. Imagine that we raise the book to a higher position that is y_f. The velocities at the both y_i and y_f are zero, which means that the change of kinetic energy is zero and the work done on the book is zero. Nevertheless, when we leave the book, it falls to the ground with a speed, which means it know posses kinetic energy. According to the work-kinetic energy theorem, there was no energy to be transferred into kinetic energy. That means that the work-kinetic energy theorem isn't applied here and there is another kind of energy rather than kinetic and work that interacts with the system. When the book was at the higher position, it had the *potential* to fall and transform some energy into kinetic energy. This energy storage mechanism is called **Potential Energy**. Potential energy depends of the position of the objects in space relative to others. Changing the positions of the objects changes the potential energy and vice versa.

To derive an equation for potential energy, we will assume that an external upward force is moving an object slowly with no acceleration upward from y_i to y_f. The ΔD in Equation 7.1 is equal to $y_f - y_i$, which transforms the equation to:

$$W_{ext} = F\Delta d = (mg)(y_f - y_i) = mgy_f - mgy_i \tag{7.7}$$

We can notice that Equations 7.6 and 7.7 are similar as both depict the external work done as the change of a form in energy. In 7.6, it was kinetic energy, but in 7.7 it is what we call potential energy. The units of potential energy is like other energy types, Joules. Also, it is a scalar quantity.

> **Remark 7.3.** The quantity mgy is the gravitational potential energy U_g of the system of earth and an object with mass m.

From our definition of gravitational potential energy and Equation 7.7, we can write the work as:

$$W_{ext} = \Delta U_g \tag{7.8}$$

7.6 Power

So, what is power that is involved in every thing in our life? In this section we will understand the concept of power and how it affects our life. So let's make a scenario that we both have a task to lift a heavy box on a table with a certain height, so I decided to lift the box vertically to the table, you decided to use a ramp with a certain angle to push the box on as you can't lift it, and another person might use a longer ramp at a smaller angle to push the box on. The Final result is the same for all of us that we all lift the box to the certain required height, doing the same work of W = mgh, but the time needed for that is different as the time needed to lift the box vertically is less than that needed using a ramp, and the time using the short ramp is also less than that required using the longer ramp. Although the work done is the same, there is something different about the tasks: the time interval during which the work is done and this is exactly the idea of power.

Definition 7.2. Power is defined as the time rate of energy transfer.

$$P = \frac{dE}{dt} \tag{7.9}$$

In this section, we will focus on work as the energy transfer method, but keep in mind that all the other methods of energy transfer, that will be discussed in the next chapter, could be used.
If an external force is applied to an object If an external force is applied to an object (which we model as a particle) and if the work done by this force on the object in the time interval Δt is W, the average power during this interval is

$$P_{avg} = \frac{W}{\Delta t} \tag{7.10}$$

Therefore, in previous conceptual example, although the same work is done in rolling the box up both ramps, less power is required for the longer ramp. In a manner similar to the way we approached the definition of velocity and acceleration, the instantaneous power is the limiting value of the average power as Δt approaches zero:

$$P = \lim_{\Delta t \to 0} \frac{W}{\Delta t} = \frac{dW}{dt} \tag{7.11}$$

where we have represented the infinitesimal value of the work done by dW. From the previously discussed information about work, We find that:

$$dW = \vec{F}.d\vec{r}$$

Therefore, the instantaneous power can be written as:

$$P = \frac{dW}{dt} = \vec{F}.\frac{d\vec{r}}{dt} = \vec{F}.\vec{v} \tag{7.12}$$

The SI unit of power is called Watt(W) after James Watt which is equivalent to J/s.

$$1W = 1J/s = 1kg.m^2/s^3$$

Another unit of power used in the US customary system is the horse-power(hp):

$$1hp = 746W$$

Chapter 8

Conservation of Energy

Contents

8.1 Conservation of mechanical energy in a non-isolated system

As previously discussed in Chapter 7, any part of the universe can be considered a single system according to a specific physical situation we want to deal with. In this chapter, we discuss two different types of systems that illustrate the concept of energy conservation.

As mentioned before, a system can be affected by the surrounding environment in many different ways. If we consider an object as a single system that is affected by an external force, this is an example of a non-isolated system. So, a non-isolated system can be defined as follows:

Definition 8.1. A non-isolated system is one in which energy can cross its boundaries, either from the surrounding environment to the system or vice versa.

There are a lot of examples in our life that help us completely understand how the energy transfer into or out of the non-isolated system happens. Imagine the money in your bank account as a non-isolated system whose balance is constant until something changes that balance. But how can this balance be changed?
While dealing with scenarios involving systems, we should always keep in

mind that energy can neither be created nor destroyed, meaning that energy is always conserved. That means if the total amount of energy in the system changes, it must be because the energy has crossed the boundaries of the system, and the total amount of energy gained is equal to the transferred energy from the surroundings.

So, by returning to the bank account example, the balance would remain conserved until changed by deposits or withdrawals.

Now if we want to express the principle of conservation of energy mathematically it will be in the following form:

$$\Delta E_{system} = \Sigma T$$

where ΔE is the change in energy, and ΣT is the total transfer of energy into or out of the system.

The transferred energy could be in the form of external Work W_{ext}, Heat energy Q, Mechanical Waves T_{MW}, Matter Transfer T_{MT}, Electrical Transmission T_{ET}, and Electromagnetic Radiation T_{ER}.

So the full expansion of the equation is:

$$\Delta K + \Delta U + \Delta E_{int} = W + Q + T_{MW} + T_{MT} + T_{ET} + T_{ER}$$

where ΔK represents the change in kinetic energy, ΔU represents the change in potential energy, and ΔE_{int} represents the change in internal energy.

Suppose that a force is applied to a non-isolated system changing its velocity and causing it to make a certain displacement. In this case, the external work done is the only form of energy transfer into the system, and the kinetic energy is the only type of energy that is affected by that work, so the conservation of energy equation,in this case,reduces to:

$$\Delta K = W_{ext}$$

This gives us the Work-Kinetic energy theorem which is a special case from the general equation.

From the previous example, we find that the general conservation of energy equation could be simplified into several special cases according to the scenario we are dealing with.

8.2 Conservation of mechanical energy in an Isolated system

For systems without nonconservative forces, mechanical energy (E_{mech}), defined as the sum of kinetic energy (K) and potential energy (U), is conserved. This principle can be expressed mathematically as:

$$E_{\text{mech}} = K + U = \text{constant}. \tag{8.1}$$

Claim 8.1 — In an isolated system, where no external forces act and no energy is exchanged with the surroundings, the total mechanical energy is conserved. This is represented by the equation:

$$K_f + U_f = K_i + U_i, \tag{8.2}$$

where the subscripts i and f denote the initial and final states of the system, respectively.

The conservation of energy is particularly useful in analyzing systems where energy transformations occur, such as converting potential energy into kinetic energy or vice versa. The following example illustrates this concept:

Example 8.1

Consider a book–Earth system. A book of mass m is lifted to a height y_i and then released from rest. Initially, the system has gravitational potential energy $U_i = mgy_i$ and no kinetic energy ($K_i = 0$). As the book falls, its potential energy decreases while its kinetic energy increases. At any point during the fall, the total mechanical energy is conserved:

$$\frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_i^2 + mgy_i. \tag{8.3}$$

This demonstrates the transformation of energy within the system due to the work done by gravity.

To further clarify, let's define the concept of mechanical energy:

Definition 8.2. Mechanical Energy is the sum of kinetic energy (K) and potential energy (U) in a system. It is given by:

$$E_{\text{mech}} = K + U, \tag{8.4}$$

where kinetic energy is expressed as $K = \frac{1}{2}mv^2$ and potential energy depends on the system, such as $U = mgh$ for gravitational potential energy.

> **Remark 8.1.**
> 1. The conservation of mechanical energy is valid only in the absence of nonconservative forces, such as friction or air resistance, which dissipate energy as heat or sound.
> 2. The principle applies universally to all isolated systems, regardless of their complexity, as long as energy exchange with the surroundings is negligible.

To explore this concept further, consider the scenario of a pendulum:

Example 8.2

A pendulum of mass m is released from a height h_i. At the lowest point of its swing, all the potential energy is converted into kinetic energy. Assuming no air resistance:

$$U_i = mgh_i, \quad K_i = 0 \quad \Rightarrow \quad K_f = \frac{1}{2}mv_f^2, \quad U_f = 0. \tag{8.5}$$

Using conservation of energy:

$$mgh_i = \frac{1}{2}mv_f^2 \quad \Rightarrow \quad v_f = \sqrt{2gh_i}. \tag{8.6}$$

This equation predicts the velocity of the pendulum at its lowest point based on its initial height.

Chapter 9

Linear Momentum and Collisions

Contents

At previous chapters, we've faced situations where we regard forces, accelerations and motion over time. We've also usually defined systems and applied the conservation of energy principle to those systems. Let us consider a different situation now.

Example 9.1

A 50-kg boy on a massless skateboard initially at rest throws a 0.035-kg rock horizontally with a speed of $60m/s$. What is the skater's speed after throwing the rock?

Analysing the situation using Newton's third law, we know that the skater will face an opposite force that will throw him away after exerting a force on that rock. The problem arises when we try to get the skater's velocity in that opposite direction. We cannot determine his speed using the kinematics equations since we do not have information about his acceleration, we cannot use force models because we do not have any information about the force of his arm on the rock, and we cannot use energy conservation because we do not have any idea bout the work done. This chapter will be allocated for introducing another quality of motion that will significantly help in solving this problem.

9.1 Linear momentum

Despite us seeing the previous problem as challenging when we consider our current models for describing motion, it is an overly simple problem if we can find another quantity that is essentially describing the quantity of motion an object has. Consider a system of two particles with masses m_1 and m_2 moving with velocities V_1 and v_2. Say that particle one is affecting particle two with some force $\vec{F}_{12}$ (gravitational force for example). From Newton's third law we know that particle two will then exert a reaction force $\vec{F}_{21}$. This action-reaction pair situation could be described mathematically through $\vec{F}_{12} = -\vec{F}_{21}$ or could be expressed as:

$$\vec{F}_{12} + \vec{F}_{21} = 0 \tag{9.1}$$

This equation just states that the sum of forces acting on particles in an isolated system is zero.
Now Let's incorporate Newton's second law into the equation noting that each particle at the moment captured is associated with an acceleration resulting from the force acting on it. Substituting $m\vec{a}$ for each force in the previous equation gives:

$$m_1\vec{a}_1 + m_2\vec{a}_2 = 0$$

Now we replace the accelerations with the definition acquired using derivatives in chapter 4

$$m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} = 0$$

Since the masses m_1 and m_2 are assumed to be constants, we bring them inside the derivatives using the constant rule:

$$\frac{d(m_1\vec{v}_1)}{dt} + \frac{d(m_2\vec{v}_2)}{dt} = 0$$

$$\frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2) = 0$$

Notice that the derivative of the sum $m_1\vec{v}_1 + m_2\vec{v}_2$ with respect to time equal zero. This mean that the sum of these two values will be the same in an isolated system at any instance. This is situation is so similar to what we had in chapter 8 about the conservation of energy in an isolated system. The conservation of the quantity $m\vec{v}$ in an isolated system led us to recognize it as a powerful tool in mechanics, especially when regarding studying the motions of objects at two distinct instants. We now call this quantity *linear momentum*.

Definition 9.1. Linear momentum $\vec{p}$ of a particles defined as the product of the particle's mass m and its velocity $\vec{v}$

$$\vec{p} = m\vec{v} \tag{9.2}$$

From equation (9.2) it could be concluded that linear momentum is a vector quantity since it equals the product of a scalar quantity m and a vector quantity $\vec{v}$. Its direction is directed along the direction of $\vec{v}$, and its SI unit is $kg \cdot m/s$

> **Remark 9.1.** Notice that any particle with momentum $\vec{p}$ would have three components:
>
> 1. $\vec{p_x} = mv_x$
> 2. $\vec{p_y} = mv_y$
> 3. $\vec{p_z} = mv_z$

As previously stated, momentum quantifies motion.This is a reasonable claim considering momentum is the product of the quantity responsible for resisting motion, mass m, and the quantity describing "how much" is the object moving, velocity v. It, also, provides a distinction between light and heavy objects moving at the same speed. A bike moving at the same speed that of a car would have way less momentum, because it has less mass. Meaning that the car is "moving more" given it is harder to move that big of a mass. This difference in the quantity of motion is apparent upon the collision of the car and the bike with any other object. The car after during the collision with a pedestrian would do damage far way more than that of the bike.

One could argue that we already have discussed a quantity that relates to the motion of an object also through mass and velocity. That quantity being kinetic energy which if you recall from equation (7.x) is equal to $\frac{1}{2}mv^2$. It is fair to question our need for another quantity that also just uses the same two quantities. Our need for momentum is based on two key differences between it and kinetic energy. First, their nature as physical quantity.since it includes squaring the velocity and consequently getting only the square of the magnitude without a direction, kinetic energy is a scalar quantity. momentum, on the other hand, is a vector quantity. In a system with two particles of same mass and speed moving in opposite directions, there is a kinetic energy associated with the system while the total momentum is zero because of its vector nature.

The second distinct future about momentum is the consistency in form with time. Kinetic energy continuously transform from a type of energy to the other, either potential energy, internal energy, or electric energy. Momentum, however, does not take on such transformation. There is only one type of linear momentum and it can not be converted to any other. Those two significant differences makes using momentum a powerful tool in independent situations from those we use energy considerations in.

Momentum is indeed tightly related to Newton's second law of motion as we noticed in the derivation above. The general form of relating the resultant of forces acting on a particle to its linear momentum can be derived if we started with newton's second law of motion and replaced the acceleration with the differential definition of acceleration:

$$\Sigma F = m\vec{a} = m\frac{d\vec{v}}{dt}$$

Since the mass in Newton;s second law is assumed to be constant, We can bring the m inside the derivative operation:

$$\Sigma F = \frac{d(m\vec{v})}{dt} = \frac{\vec{p}}{dt} \tag{9.3}$$

This equation states that the time rate of change of the linear momentum of a particle is equal to the net force acting on the particle.

9.2 Isolated system(Momentum)

Recalling from the previous section, in our setup for particles we reached to equation (9.1) that can be written in another form:

$$\frac{d}{dt}(\vec{p_1} + \vec{p_2}) = 0$$

Since the change of the total momentum $\vec{p_{tot}} = \vec{p_1} + \vec{p_2}$ is zero, we can say that the total momentum is constant or more conveniently that over a time interval the change in the total momentum is zero:

$$\Delta \vec{p_{tot}} = 0 \tag{9.4}$$

More frequently, you will use the form of the equation with every momentum present:

$$\vec{p_{1i}} + \vec{p_{2i}} = \vec{p_{1f}} + \vec{p_{2f}}$$

Where $\vec{p_{1i}}$ and $\vec{p_{2i}}$ are the initial momenta of particles 1 and 2, while $\vec{p_{1f}}$ and $\vec{p_{2f}}$ are their final momenta.

Remark 9.2. 1. Notice that the equation is still applicable on all dimensions separately:
- $\vec{p_{1ix}} + \vec{p_{2ix}} = \vec{p_{1fx}} + \vec{p_{2fx}}$
- $\vec{p_{1iy}} + \vec{p_{2iy}} = \vec{p_{1fy}} + \vec{p_{2fy}}$
- $\vec{p_{1iz}} + \vec{p_{2iz}} = \vec{p_{1fz}} + \vec{p_{2fz}}$

2. The equations can be extended by any number of particles

In words, an the interactions of momentum in an isolated system can be described through the statement:

Claim 9.1 — Whenever two or more particles interact within an isolated system, the total momentum of the system remains constant.

Now if we Consider the problem from earlier:

Example 9.2

A 50-kg boy on a massless skateboard initially at rest throws a 0.035-kg rock horizontally with a speed of $60m/s$. What is the skater's speed after the throwing the rock?

Solution. We start by realizing that the only motion we are dealing with is horizontal motion, so we will donate for the x component of momentum by just momentum. We refer for the boy as particle 1 and the rock as particle 2.

From equation (9.4), we note that the total momentum before and after throwing the rock is the same.

$\Delta\vec{p} = 0 \rightarrow \vec{p_f} - \vec{p_i} = 0 \rightarrow \vec{p_f} = \vec{p_i}$

$$(m_1\vec{v_{1i}}) + (m_2\vec{v_{2i}}) = (m_1\vec{v_{1f}}) + (m_2\vec{v_{2f}})$$

Notice that the initial both speeds of the boy and the rock initially are zero, so the terms on the right side cancels.

$$(m_1\vec{v_{1f}}) + (m_2\vec{v_{2f}}) = 0$$

Now we solve for the velocity of the skater:

$$\vec{v_{1f}} = -\frac{m_2}{m_1}\vec{v_{2f}} = -(\frac{0.035kg}{50kg})(60m/s) = -0.042m/s$$

The negative sign of the velocity means that the skater will move in the direction opposite to that of the rock. ■

9.3 Non isolated system (Momentum)

In chapter 8 we discussed energy in isolated an non-isolated systems. Similarly, that is what we will be doing here but considering momentum instead. When discussing non isolated systems in chapter 8, we noted that the system's energy changes if energy passes through the boundaries of the system in or out. The mechanism of transferring momentum in or out of a system, however, is a net external force from the environment acting on the system fro an interval of time.

To enhance our comprehension of this, let us consider a setup where a net external force $\Sigma\vec{F}$ is acting on a system consisting of one particle. Note that the force could Vary with time and is not necessary constant. From newton's second law's general form, equation (9.3), we can re arrange the variables by multiplying dt to both sides:

$$d\vec{p} = \Sigma\vec{F}dt$$

We can integrate both sides of the equation to get the total change of momentum $\Delta\vec{P_{tot}} = \vec{p}_f - \vec{p}_i$ over the time interval $\Delta t = t_f - t_i$:

$$\Delta\vec{P_{tot}} = \vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \Sigma\vec{F}dt$$

The quantity on the right side of the equation is called *Impulse*:

$$\vec{I} = \int_{t_i}^{t_f} \Sigma\vec{F}dt \tag{9.5}$$

Definition 9.2. Impulse is a vector quantity having a magnitude equal to the area under the force-time curve and a direction same as that of the change in momentum vector. It has the dimension ML/T and the unit $N \cdot s$.

> **Remark 9.3.** Impulse is not a property of a particle but a measure of the change in momentum caused by and external force.

The previous equation accounts for variations of force over the time interval, but to get a simpler form we can define a value for the average net force over a time interval $\Delta t = t_f - t_i$:

$$\frac{\text{All Values of } \Sigma F}{\text{The time interval}} = \frac{\int_{t_i}^{t_f} \Sigma\vec{F}dt}{\Delta t}$$

$$(\Sigma F)_{avg} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \Sigma\vec{F}dt \tag{9.6}$$

Following this, we could express equation (9.5) as:

$$\vec{I} = (\Sigma F)_{avg} \Delta t \tag{9.7}$$

Combining all the previous we can reach a fundamental statement in mechanics that is **impulse-momentum theorem**:

> **Claim 9.2 —** The impulse applied to an object will be equal to the change in its momentum:
> $$\Delta \vec{p} = \vec{I}$$

9.4 collisions

In this section, we will use our understanding of momentum in an isolated system to discuss the simplest application of the conservation of momentum principle, **collisions**. The term *collision* refers to the interaction of two or more by forces particles that come very close to each other.

A collision typically involve "physical contact". Consider a simple game of billiards: when a cue ball strikes another ball, both balls move in new directions, exhibiting changes in speed and trajectory. However, the concept of collisions must be made clear and refined to include some other scenarios with what is seen as no "physical contact". Consider a proton and an alpha particle moving opposing to each other. Since the two particles are positively charged, they repel each other when they come sufficiently close. Thus, they do interact with pairs of forces, and this interaction should be considered as a collision. To predict the outcomes of both cases accurately, we will need to understand the principles of momentum in the context of collisions.

Whenever two particles of masses m_1 and m_2 collide together as expressed in any of the two cases above, repulsive forces occur, and in a more realistic scenario these forces vary with time in complex trends. However complex these forces are, they are still internal to the system including of the two particles making it an isolated system. Thus, the momentum is *always* conserved in a collision. This is not necessarily true for kinetic energy, however. collisions are actually divided into **elastic** and **inelastic** collisions according to whether or not is kinetic energy conserved.

9.4.1 Inelastic Collisions

Inelastic collisions are those in which kinetic energy is not the same after the interaction. In an **inelastic collision** the momentum is conserved after the two objects collide but some of the kinetic energy is lost through heat, sound, or internal energy causing deformations. Another extreme case is a **perfectly inelastic collision**, in which the two bodies become one after the collision (an arrow sticking on a target or a meteorite hitting the earth are two examples).

consider two particles of masses m_1 and m_2 moving with initial velocities $\vec{v_{1i}}$ and $\vec{v_{2i}}$ along the same line but opposing to each other. The two particles collide and stick together to finally move with a common velocity $\vec{v_f}$. Since the momentum in such an isolated is always conserved, we can state that the total momentum before the collision is the same after:

$$\Delta\vec{p} = 0 \rightarrow \vec{p_i} = \vec{p_f} \rightarrow m_1\vec{v_{1i}} + m_2\vec{v_{2i}} = (m_1 + m_2)\vec{v_f}$$

Solving for the final velocity gives:

$$\vec{v_f} = \frac{m_1\vec{v_{1i}} + m_2\vec{v_{2i}}}{m_1 + m_2} \tag{9.8}$$

9.4.2 Elastic Collisions

Opposing to inelastic collisions, in elastic collisions the total kinetic energy is the same before and after the collision, no energy is lost through heat, sound, or internal energy and deformations. From its definition it gets kind of clear that elastic collisions are impossible on the macroscopic level. Imagine the billiard ball collision from earlier. You can indeed hear the sound of the collision and know that energy transferred through this sound out of the system. sometimes such collisions are just assumed to be *approximately* elastic. Elastic collisions are more frequent on atomic and sub atomic levels, between protons, photons, and electrons.

Consider two particles with masses m_1 and m_2 moving with initial velocities $\vec{v_{1i}}$ and $\vec{v_{2i}}$ along a straight line opposing to each other then rebound along the same line after the collisions with speeds $\vec{v_{1f}}$ and $\vec{v_{2f}}$. In elastic collisions the momentum and kinetic energy are conserved as in:

$$\vec{p_i} = \vec{p_f} \to m_1\vec{v_{1i}} + m_2\vec{v_{2i}} = m_1\vec{v_{1f}} + m_2\vec{v_{2f}} \tag{9.9}$$

$$K_i = K_f \to \frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{1f}^2 \tag{9.10}$$

Remark 9.4. There are some known forms obtained by solving equations (9.9) and (9.10) simultaneously including:

1. $v_{1i} + v_{1f} = v_{2i} + v_{2f}$
2. $v_{1f} = (\frac{m_1-m_2}{m_1+m_2})v_{1i} + (\frac{2m_2}{m_1+m_2})v_{2i}$
3. $v_{2f} = (\frac{2m_2}{m_1+m_2})v_{1i} + (\frac{m_2-m_1}{m_1+m_2})v_{2i}$

9.4.3 Special Cases

Let us consider some special cases of collisions. If $m_1 = m_2$, the equations in remark (9.4) show that then $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$, which means that the two particles just exchange velocities when they are of the same mass. That is apparent in the well known newton's cradle.

Chapter 10

Rotation of a Rigid Object About a Fixed Axis

Contents

10.1 Angular and Translational Quantities

So far we have discussed the motion of bodies in terms of translational motion. In other words, the motion where an object moves from point A to point B. The typical physical quantities associated with this analysis model are: displacement, velocity, and acceleration. In this chapter, we develop a new approach to study another sort of motion of bodies: **rotation**.

10.1.1 Angular Quantities

Consider a disc with a point on its surface. If the disc starts rotating, the point moves with it whether clockwise or anticlockwise, either way, the position of the point in this case is said to be the angle in which the point makes with some arbitrary axis usually taken to be the positive x-axis. Since the angle changes as time goes, the point has to be moving with some average **angular velocity** which is equal to $\omega_{avg} = \frac{\Delta\theta}{\Delta t}$. In order to calculate the angular velocity at any given moment we take the limit as $\Delta t \to 0$:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{10.1}$$

Since the point can be speeding up or slowing down, it can move with some average **angular acceleration** which is equal to $\alpha_{avg} = \frac{\Delta\omega}{\Delta t}$. Same goes here to calculate the instantaneous angular acceleration:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \tag{10.2}$$

It is worth noting that all points on a rotating object move with the same angular velocity and acceleration but differ in their translational velocities and accelerations.

10.1.2 Rigid Object under Constant Angular Acceleration

The kinematic equation developed in translational motion apply in rotational motion with switching every translational quantity with its rotational counterpart. The new equations are as follows:

$$\omega_f = \omega_i + \alpha t \tag{10.3}$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \tag{10.4}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \tag{10.5}$$

$$\theta_f = \theta_i + \frac{1}{2}(\omega_f + \omega_i)t \tag{10.6}$$

10.1.3 The Relationship between Translational and Rotational Quantities

Before we try and find the mathematical relationships, we must revise some geometrical rules. Consider a circle having some radius r and intercepting an angle θ between two radii which faces an arc of length s. Geometry tells us that the angle, measured in radians, is equal to the ratio between the arc length and the radius of the circle:

$$\theta(rad) = \frac{s}{r}$$

Now let's consider the translational motion of a point on a rotating object. The point moves along an arc of a circle centered at the axis of rotation. The instantaneous velocity of the point is simply the derivative of the displacement or the arc length, if taken on tiny portion of the arc. Since the arc length s can be replaced with the intercepted angle θ times the radius r:

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$v = r\omega \tag{10.7}$$

A similar approach gives the relation between the acceleration and its angular form:

$$a = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$a = r\alpha \tag{10.8}$$

In conclusion, the relationship between the translational and rotational quantities are that the translational quantity equals its rotational form times the radius.

Back to the point on a rotating object, the point can be assumed to be moving in a circular motion and hence experiencing centripetal acceleration. We can find the centripetal acceleration in terms of angular velocity instead of translational velocity as follows:

$$a_c = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = r\omega^2 \tag{10.9}$$

Example 10.1

A wheel is rotating with an angular acceleration of $\alpha = 5rad/s^2$ and angular velocity of $\omega = 50rad/s$. Calculate the velocity of a point at $0.25m$ from the center of the wheel after $10s$ have passed

Solution. We start by finding the angular velocity of the wheel after $10s$ have passed using equation 10.3:

$$\omega_f = \omega_i + \alpha t = 50rad/s + 5rad/s^2 \times 10s = 100rad/s$$

Now, to calculate the velocity we use equation 10.7:

$$v = r\omega = 0.25m \times 100rad/s = 25m/s$$

■

10.2 Torque

As we have discusses earlier, the way objects accelerate is by applying external net force on them. According to Newton's second law of motion, the net force applied to an object is directly proportional to the acceleration of the object according to the expression:

$$\sum F = ma$$

In this section, we develop a new way to use Newton's second law in rotational motion by using a new quantity: **torque**. Torque can be assumed to be the angular counterpart of the force. However, it is not equal to it multiplied by the radius as the rest of the quantities.

10.2.1 How a Tangential Force Cause a Torque

Consider a wrench rotating about a pivot point. If a force acts on the handle in some direction that is not along the wrench itself, it rotates about the pivot point and accelerates. The cause of this rotation is a torque that is caused by the force acting on the handle. Simply, The torque vector is the cross product between the displacement vector from the pivot point to the point of application and the force vector:

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10.10}$$

As a result of this equation, the magnitude of the torque vector can be calculated as follows:

$$\tau = rF\sin(\phi) = Fd \tag{10.11}$$

where d is called the **moment arm** the shortest distance between the pivot point and the point of application. In other words, it is the length of the perpendicular line connecting the pivot point and some point on the line action of the force. ϕ is the angle between the force vector and displacement vector.

If several forces acts on an object at different points, the magnitude of the net torque is the sum of the individual torques of each force:

$$\sum \tau = \sum_i F_i d_i \tag{10.12}$$

10.2.2 Bodies Under a Net Torque

Now consider a rotating object. To calculate the net torque on the object, we need to calculate the torque due to each small segment of the object and then sum the results. Since the displacement from the pivot to the point of application will always be the shortest:

$$\sum \tau = \sum_i F_i d_i = mar = m(\alpha r)r = (mr^2)\alpha = I\alpha$$

$$\sum \tau = I\alpha \tag{10.13}$$

where I is a quantity called the moment of inertia.

Equation 10.13 is the rotational counter part of Newton's second law of motion and the base of the **body under a net torque model**.

Example 10.2

A waterwheel is rotating with moment of inertia $I = 120kg \cdot m^2$ and angular velocity $\omega = 10rad/s$. If there is an external torque of $\tau = 30N \cdot m$, what is the angular velocity of the wheel after $15s$ have passed?

Solution. We first use equation 10.13 to calculate the angular acceleration;

$$\sum \tau = I\alpha \rightarrow \alpha = \frac{\sum \tau}{I} = 0.25rad/s^2$$

Using equation 10.3:

$$\omega_f = \omega_i + \alpha t = 10 + 0.25 \times 15 = 13.75rad/s$$

■

10.2.3 Calculation of Moment of Inertia

Different shapes have different moments of inertia. For instance, the moment of inertia of a rod rotating on its center is found to be $\frac{1}{2}ML^2$ where the moment of inertia of a solid sphere is $\frac{2}{5}MR^2$. The location of the axis of rotation changes the moment of inertia. For example, the moment of inertia of a rod rotating about one end is different from the one mentioned before and is found to be $\frac{1}{3}ML^2$.

In this subsection, we develop a way of calculating the moment of inertia of any shape about any axis or rotation. First, let's start with regular shapes. According to the proof of equation 10.13, The moment of inertia is the sum of the quantity $m_i r_i^2$ for each small segment. If we take the limit as the mass of the segment approaches 0 we reach the following equation:

$$I = \lim_{\Delta m \to 0} \sum_i r^2 \Delta m = \int r^2 dm \tag{10.14}$$

where r is the distance between the axis of rotation and the segment and is usually given as a function of mass to use the previous equation.

Sometimes it can be hard to calculate the moment of inertia of an object using equation 10.14. If the object is uniform, has a constant density value, we can use the relation $\rho = \frac{dm}{dv}$ to get the following equation:

$$I = \int \rho r dV \tag{10.15}$$

Some modifications can be made to the equation if the object can be assumed to have a linear mass density, denoted by λ, or surface mass density, denoted by σ rather than a volume mass density. The moment of inertia can be calculated in this case by converting the ρ in the equation by whatever equivalent it has.

Example 10.3

Calculate the moment of inertia of a uniform solid cylinder of length ℓ, radius R and mass M rotating about its central axis.

Solution. To calculate the moment of inertia using equation 10.14, we need to find a meaningful expression for the quantity dV:

$$V = \pi R^2 \ell \rightarrow \frac{dV}{dr} = 2\pi r \ell \rightarrow dV = 2\pi r \ell dr$$

Now by substituting the value into the equation:

$$I = 2\pi\rho \int_0^R r^3 dr = \frac{1}{2}\pi\rho\ell R^4$$

Substituting the value of the density:

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 \ell} \rightarrow I = \frac{1}{2} M R^2$$

■

10.3 Energy Considerations in Rotational Motion

As usual, several problems in rotational motion can be easily solved using torque considerations. Nonetheless, another useful way is using energy considerations which allows for easy solutions most of the time.

10.3.1 Rotational Kinetic Energy

A rotating object has some rotational kinetic energy but no translational one since there is no linear velocity. To find the rotational kinetic energy of a body, consider a rotating rigid body which we divide to small segments of masses Δm. The rotational kinetic energy of the total body is the sum of the translational kinetic energies of the individual segments as follows:

$$K_R = \sum K_i = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

$$K_R = I\omega^2 \tag{10.16}$$

10.3.2 Work in Rotational Motion

To calculate the work done on a rotating object, we have to start first, as usual, by the translational quantities. Consider a rotating rigid object and there is a force F acting on some point at a distance r from the pivot point where it moves a displacement of ds along the arc of the circle centered at the pivot. According the equation of work:

$$W = \vec{F} \cdot \vec{r} \rightarrow dW = (F \sin(\phi)) r d\theta$$

Since the quantity $Fr\sin(\phi) = \tau$:

$$dW = \tau d\theta \tag{10.17}$$

Taking the rate of the change of work and angle with respect of time gives:

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt} \rightarrow P = \tau\omega \tag{10.18}$$

Now to find the exact value of the work done on a body, we need to develop some relationship first as follows:

$$\sum \tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta} \rightarrow \sum \tau d\theta = I\omega d\omega$$

Integrating the previous equation gives:

$$W = \int \sum \tau d\theta = \int_{\omega_i}^{\omega_f} I\omega d\omega = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2 = \Delta K_R \tag{10.19}$$

Chapter 11

Angular Momentum

Contents

Previously, we introduced the concept of a linear force, which led to the translational motion of an object. A more advanced concept was the torque, which led to the rotation of an object. In Chapter 9, we used linear momentum to describe the translational motion of an object. In this chapter, we will introduce a more advanced concept that describes the rotation of an object.

11.1 Angular Momentum of a Particle and an Object

Consider a particle of mass m with position vector $\vec{r}$ and linear momentum $\vec{p}$. Accordingly, the sum of forces acting on this particle is equal to the first derivative of $\vec{p}$ with respect to time.

$$\sum_i \vec{F} = \frac{d\vec{p}}{dt} \tag{11.1}$$

Cross-product both sides of the equation with the vector $\vec{r}$.

$$\vec{r} \times \sum_i \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} \tag{11.2}$$

The left-hand side of the equation reduces to the sum of torques acting on the object. For the right-hand side, add $\frac{d\vec{r}}{dt} \times \vec{p}$. Because $\frac{d\vec{r}}{dt} = \vec{v}$, and $\vec{p}$ and $\vec{v}$

have the same direction, their cross product is just a zero.

$$\sum_i \vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p} \tag{11.3}$$

Notice that the left-hand side of the equation is just the first derivative of $\vec{r} \times \vec{p}$ with respect to time.

$$\sum_i \vec{\tau} = \frac{d(\vec{r} \times \vec{p})}{dt} \tag{11.4}$$

Notice the analogy between the Equation 11.4 and Equation 11.4. Because torque leads to the rotational motion of an object, as opposed to the force that leads to its translational, the product $\vec{r} \times \vec{p}$ should substitute the term $\vec{p}$ that describes translational motion. Thus, $\vec{r} \times \vec{p}$ describes the object's rotational motion, which we call angular momentum and give it the symbol $\vec{L}$.

Definition 11.1.1 The angular momentum $\vec{L}$ of a rotating particle is given by the cross product of $\vec{r}$ and $\vec{p}$, where $\vec{r}$ is the position vector of the particle, $\vec{p}$ is its linear momentum.

$$\vec{L} = \vec{r} \times \vec{p} = \|\vec{p}\|\|\vec{r}\| \sin(\theta) = mvr \sin(\theta) \tag{11.5}$$

where θ is the angle between $\vec{r}$ and $\vec{p}$. Thus, the net torque on the particle can be given by

$$\sum_i \vec{\tau} = \frac{d\vec{L}}{dt} \tag{11.6}$$

Equation 11.6 can be rewritten in such a way to give the rotational version of the *impulse-momentum theorem*, the *angular impulse-angular momentum theorem*.

Definition 11.1.2 The angular impulse-angular momentum theorem states that the angular impulse (the change in angular momentum) can be given by

$$\mathit{\Delta}\vec{L} = \int \left(\sum_i \vec{\tau} \right) dt \tag{11.7}$$

Since $L = mvr \sin(\theta)$, we can deduce its SI unit to be $kg \cdot m^2 \cdot s^{-1}$

Example 11.1

A particle of mass 5 kg is rotating around a circle with radius 2 m. Find its angular momentum if the centripetal force acting on the particle equals 40 N.

Solution. We start by calculating the particle's velocity.

$$F_c = \frac{mv^2}{r} \implies v = \sqrt{\frac{rF_c}{m}} = \sqrt{\frac{2 \cdot 40}{5}} = 4 \ m/s$$

Recall that when the particle is under a centripetal force, its velocity is perpendicular to the radius of the circle it is rotating around. Thus, the angle between the position of the particle $\vec{r}$ and its momentum $\vec{p}$ is $\frac{\pi}{2}$. Accordingly, the particle's angular momentum is

$$L = mvr\sin(\theta) = 5 \cdot 4 \cdot 2 \cdot \sin(\frac{\pi}{2}) = 40 \ kg \cdot m^2 \cdot s^{-1}$$

■

What if we are finding the angular momentum of a rotating rigid object? Notice that every particle of the rigid object rotates around the axis the object is rotating around. For simplicity, we will only consider the case where the position vector $\vec{r}$ is perpendicular to the angular momentum $\vec{p}$, i.e the angle between them is $\frac{\pi}{2}$. For the i^{th} particle, the angular momentum is

$$L_i = m_i v_i r_i = m_i r_i^2 \omega \tag{11.8}$$

Notice that ω is fixed for all particles. Summing over i to get the total angular momentum of the object yields

$$L = \sum_i m_i r_i^2 \omega = \left(\sum_i m_i r_i^2 \right) \omega \tag{11.9}$$

Recall the formula for the moment of inertia $I = \sum_i m_i r_i^2$. Accordingly, we can reach the following simple formula.

Definition 11.1.3 For a rigid rotating object about an axis, the angular momentum can be calculated as follows

$$L = I\omega \tag{11.10}$$

where I is the moment of inertia, ω is the angular velocity.

11.2 Conservation of Angular Momentum

In Section 11.1, we discussed the angular momentum of a particle in a non-isolated system, meaning that there is a net torque acting on the system. In this section, we discuss angular momentum in an isolated system or when the net torque acting on the system is zero. To reach the main finding of this section, substitute $\sum_i \vec{\tau}$ to be zero in the angular impulse formula.

$$\Delta\vec{L} = \int \left(\sum_i \vec{\tau}\right) dt = \int 0 \; dt \tag{11.11}$$

$$\implies \Delta\vec{L} = 0 \tag{11.12}$$

Equation 11.11 means that the change in angular momentum is zero. In other words, *angular momentum is conserved in an isolated system.*

Definition 11.2.1 In an isolated system, angular momentum is conserved, meaning that its magnitude and direction are constant.

$$\Delta\vec{L} = 0 \implies \vec{L}_i = \vec{L}_f \tag{11.13}$$

11.3 Gyroscopic Effects

When you through a spin top, you notice that the top rotates about its axis of symmetry. However, if the top is spinning rapidly, you will notice that its axis of symmetry is rotating about some axis. This motion is known as precessional motion, which is to be illustrated using the concept of a gyroscope.

Consider the gyroscope in **Figure 11.1**. Two forces are acting on this gyroscope. The first one is the normal force acting on the pivot point O, while the second one is the gravitational force acting on the center of mass M. If we let the distance between M and O be r, we could say that the gravitational force produces a torque mgr. Because this torque is due to the cross product of the gravitational force and $\vec{r}$, the torque produced is perpendicular to both of these vectors. Furthermore, notice that, normally, the direction of the axis that the gyroscope rotates about which is the direction of its angular momentum is in the same direction of $\vec{r}$. Thus, the direction of the torque is perpendicular to the direction of the gyroscope's angular momentum. Recall that Equation 11.6 means, in a time interval dt, the existence of a torque $\vec{\tau}$ implies the existence of a change in the angular momentum $d\vec{L}$ in the same direction of $\vec{\tau}$. Thus, $d\vec{L}$ is perpendicular to the direction of $\vec{L}$. Because of the vector nature of angular momentum, that change in angular momentum is a change in its *direction* and not its *magnitude.*

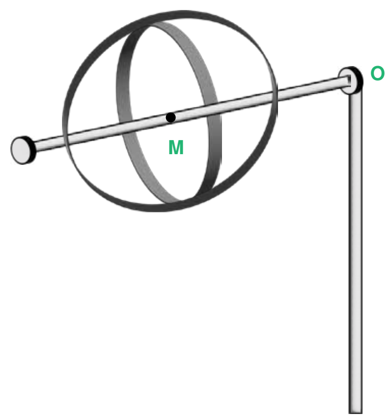


Figure 11.1: A Typical Gyroscope. O represents the pivot point. M represents the center of mass.

For simplicity, we will only take the sum of angular momentum $I\vec{\omega}$ due to the spinning and the angular momentum due to the motion of the center of mass about the pivot, neglecting the contribution from the center of mass, and taking the total angular momentum to be $I\vec{\omega}$. In practice, this is a good approximation of $\vec{\omega}$ is made large enough. On applying a torque $\vec{\tau}$, the angular momentum $\vec{L}_i$ will change to $\vec{L}_f$, where this change is only in the direction (i.e$\left\|\vec{L}_i\right\| = \left\|\vec{L}_f\right\| = L$).

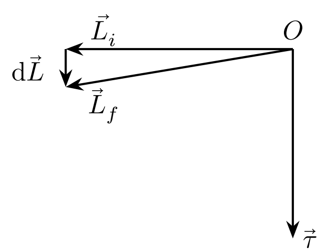


Figure 11.2: The vector diagram illustrating the change in angular momentum. Notice that $d\vec{L}$ is in the same direction as $\vec{\tau}$.

Let the angle produced by this change be $d\theta$. Recalling the length of an arc formula and utilizing the vector diagram in **Figure 11.2** yields

$$s = \theta r \implies \theta = \frac{s}{r} \implies d\theta = \frac{dL}{L} \tag{11.14}$$

Notice that dL is infinitesimally small, allowing us to approximate it as the length of an arc of a circle centered at O.

$$d\theta = \frac{dL}{L} = \frac{\sum_i \tau \ dt}{L} = \frac{(mgr)dt}{L} \tag{11.15}$$

$$\implies \omega_p = \frac{d\theta}{dt} = \frac{mgr}{I\omega} \tag{11.16}$$

where the angular speed ω_p is known as the precessional frequency, which describes the rotation of the axis of symmetry of the gyroscope about the other axis. Note that this result is valid only if $\omega >> \omega_p$ or when the gyroscope is spinning rapidly.

11.4 Rotational Kinetic Energy

Because kinetic energy is the energy associated with motion, it is only convenient that the *rotational* kinetic energy is associated with rotational motion. Consider a system of particles (a rigid body) rotating around some axis with angular velocity ω. For the i^{th} particle, the kinetic energy is

$$K_i = \frac{1}{2}mv^2 = \frac{1}{2}m_i r_i^2 \omega^2 \tag{11.17}$$

Summing over all i gives the kinetic energy of the whole system.

Definition 11.4.1 The rotational kinetic energy of a rotating rigid body can be found by

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2}I\omega^2 \tag{11.18}$$

Notice that this is not a new form of energy. This is just an alternate formula for kinetic energy that is convenient with rotational motion.

Example 11.2

A disk of mass 12 kg and radius 2 m is rotating with an angular velocity of 4 rad/s. What is the rotational kinetic energy associated with the rotation of the disk?

Solution. Recall the formula for a disk's moment of inertia.

$$I = \frac{1}{2}mr^2 = \frac{1}{2} \cdot 12 \cdot 2^2 = 24\ kg \cdot m^2$$

Accordingly, the rotational kinetic energy of the disk is

$$K_R = \frac{1}{2}I\omega^2 = \frac{1}{2} \cdot 24 \cdot 4^2 = 192\ J$$

Chapter 12

Static Equilibrium and Elasticity

Contents

After studying the motion of rigid bodies in the previous chapters, we will study the static of rigid bodies—or what we can call "Equilibrium". Equilibrium state is a common topic in many fields in Engineering applied sciences. After studying the rigid objects' statics, we will move to the Elasticity of the rigid objects and closely oversee how those objects deform and reshape under different forces.

12.1 Rigid objects in equilibrium

Equilibrium is what describes an object when it has constant or zero motion. Before, we had investigated the Equilibrium of a particle and stated that the particle should has a constant velocity. This implies that the Total Forces (Net Force) is equal to zero. For a real object with different dimensions (not a particle), another term must be set to zero to achieve the desired Equilibrium: angular acceleration. As we know from Equation XX

$$\sum \tau = I\alpha$$

to have a constant angular velocity, we must have a zero angular acceleration, and the Net angular Force (Torque) must be zero. That implies that the sum of Fd should be equal to zero. This situation leads as to a new term: Rotational Equilibrium.

From the previous, we can conclude the conditions of a static Equilibrium of a rigid body:

1. $$\sum F = 0$$

2. $$\sum \tau = 0$$

> **Remark 12.1.** The equilibrium of rigid bodies can occur only if both the translational and rotational forces are equal to zero, leading the linear and angular velocities to be constant or zero.

When dealing with a body in the XY-plane, we deal with three equations. The first two are the Net Forces in the X and the Y directions. The third is the Net torque in the Z direction. In 3-D system, there will be six equations with many unknowns (three for the force in the 3 dimensions and three for the torque). Although the 3-D system is the general one, the 2-D is the easier to solve.

12.2 Center of Gravity

When a rigid body problem arises, the acting point of the force must be known. The acting point is the point where the force would be if the rigid object is dealt with as a particle. If we didn't assume this shortcut, we have to find the force that acts in each infinitesimal point, which is impossible. When the force of gravity acts on a body, we must determine the point that making the same force on it will result in the same result, which is called Center of Gravity.

> **Remark 12.2.** The center of gravity is the point where the mass appears to be concentrated at, or the point where the force of gravity appears to act upon the rigid object.

Now, we need to find this Center of Gravity (CG) point. Firstly, we will assume that the gravity acts uniformly with equal magnitude on the rigid body. As we had said, the effect of the forces on the object equals the effect on the center of gravity point. The effect that we will consider is the Torque. The magnitude of an individual torque will equal

$$Fd = mgd$$

The total torque value will be the sum of the individual torques:

$$m_1 g_1 d_1 + m_2 g_2 d_2 + ...$$

When considering the CG of the body, the x position will be x_{CG}, and g will be g_{CG}, indicating that the body is acting as a point that equals the total mass. Then, the equation of the total torques will be

$$m_1 g_{CG} d_{CG} + m_2 g_{CG} d_{CG} + ...$$

Equalizing the two equations, we will get

$$m_1 g_1 d_1 + m_2 g_2 d_2 + ... = m_1 g_{CG} d_{CG} + m_2 g_{CG} d_{CG} + ...$$

and since the gravity is uniform on this object, $g_1 = g_2 = g_{CG}$ and they cancel each other. The equation then transforms to:

$$X_{CG} = \frac{\sum mx}{\sum m}$$

giving the same final value as the center of mass, considering the gravity is uniform on the object.

12.3 Elastic properties of Solids

When a solid object undergoes the application of an external force, deformation happens to the object–to some extent–while an internal force balances the external one, resisting the deformation. The ratio between that internal force and the area acted upon is defined as *stress*. **Stress** is the internal restoring force–a response to the external force–of the object per cross-sectional area unit. The measure of the resulting deformation is a quantity known as *strain*. For sufficiently small stresses, it was found, through experiments, that stress is proportional to strain through a proportionality constant known as the **elastic modulus** specific to the material being deformed AND* the nature of deformation.

$$\text{Elastic modulus} = \frac{\text{Stress}}{\text{Strain}} \tag{12.1}$$

Definition 12.1. **elastic modulus* is generally the ratio between what is being done (action) to the object to how it responds (reaction).

> **Remark 12.3.** There are *three* types of deformation, and we define the elastic modulus for each of them:
>
> 1. **Young's modulus**: a measurement of the resistance of an object to a change in its *length*.
> 2. **Shear modulus**: a measurement of the resistance of an object to the *motion* of the planes parallel to each other within the object.
> 3. **Bulk modulus**: a measurement of the resistance of an object to a change in its *volume*.

12.3.1 Young's Modulus

Consider a steel bar, fixed from one end with length L_i and cross sectional area of A (Figure 12.1). When it is acted upon with external force $\vec{F}$ that is perpendicular to the plane of A, as a result, the bar reaches "equilibrium" in which the internal molecular forces resist distortion and balance out the external forces, while the final length of the bar becomes L_f that exceeds L_i by ΔL–as the bar is said to be *stressed*. **Tensile stress** is defined to be the magnitude of the ratio between the external force $\vec{F}$ and the cross sectional area A, while the **tensile strain** is defined to be the magnitude of the ratio between ΔL and L_i. Consequently, **Young's modulus** is set to be the ratio between *tensile stress* and *tensile strain*.

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{\Delta L/L_i} \tag{12.2}$$

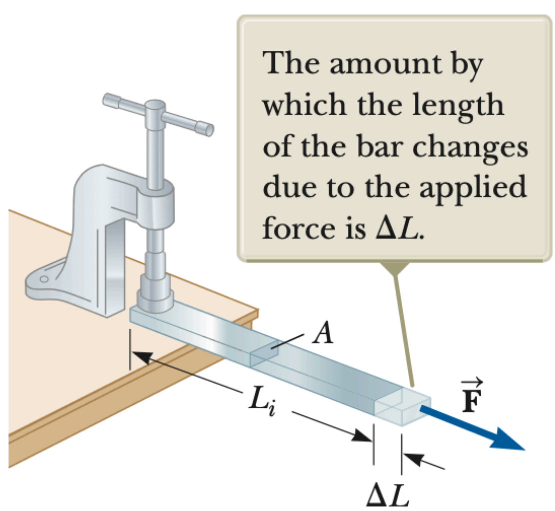


Figure 12.1: A steel bar undergoing stress resulting in a change in its length

Claim 12.1 — For small enough stresses, that deformation is NOT permanent; the bar rebounds to its initial length as long as the stress value does not hit the elastic limit, which is the maximum stress value at which the material possess the elasticity property (i.e. beyond that value, the object would have some sort of permanent distortion). The elastic behaviour naturally happens in correlation with the amount of stress until the stress reaches the elastic limit, beyond which the elastic behaviour is no longer in correlation and tends to be lower until the **breaking point**.

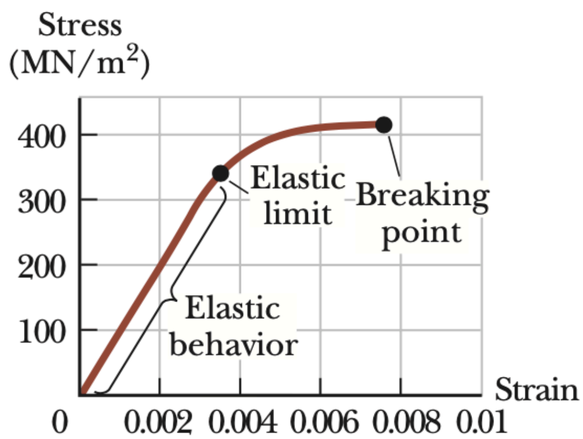


Figure 12.2: Stress-Strain curve for elastic solid object

12.3.2 Shear Modulus

Suppose a rectangular block is acted upon two parallel opposite forces: $\vec{F}_1$ is applied on one face, while the opposite $\vec{F}_2$ is holding the opposite face. The stress resulting from the actions of the applied forces is called *shear stress*, which is defined as F/A–the ratio of the force component coplanar with the side face to the cross sectional area A of the face being sheared. *Shear strain* is defined to be the ratio between the distance the sheared face moved, Δx, to the height of the rectangular block h. Thus, we define the **Shear modulus** to be the ratio between shear stress to shear strain.

$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h} \tag{12.3}$$

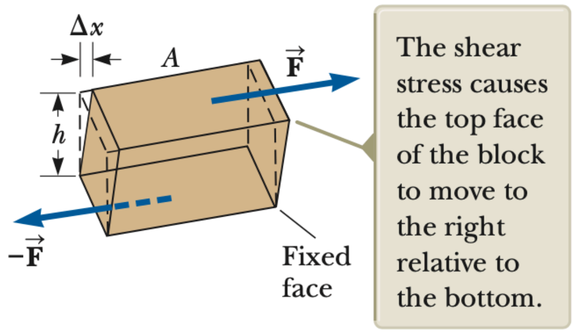


Figure 12.3: Shear stress illustration

12.3.3 Bulk Modulus

As previously mentioned, the Bulk modulus is associated with the object's resistance to changes in its volume. A change in volume typically occurs under the effect of a net force uniformally distributed along the surface area of the object, which results in a change in volume and not in shape. Such a situation usually happens when the object is immersed within a fluid, as we will see in chapter 14. *Volume stress* is the ratio between magnitude of the uniformally distributed force F to the surface area A upon which the force is applied. As the object is immersed within a fluid, it experiences *pressure*, which is defined to be $P = F/A$ (explored in-depth in chapter 14). If pressure changes, so does the volume of the object by an amount of ΔV. *Volume strain*–of course you guessed it right–is the ratio between the change in volume ΔV and the initial volume V. Not surprisingly, the Bulk modulus is defined to be the ratio between the volume stress to the volume strain. As shown in **figure 12.4**, when a cube, with volume V_i, is *pressurized* (i.e. acted upon with uniformally distributed forces on the surface area), it undergoes a change in volume WITHOUT a change in shape.

$$B = \frac{\text{volume stress}}{\text{volume strain}} = \frac{F/A}{\Delta V/V_i} \tag{12.4}$$

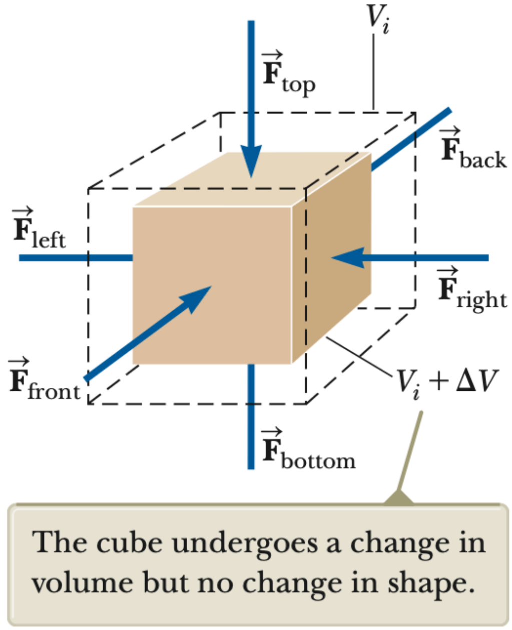


Figure 12.4: Volumetric stress illustration

Chapter 13

Universal Gravitation

Contents

13.1 Newton's law of gravitation

Yes, it's the apple one. I'm sure you have flashbacked the story from your elementary school: "One day, Mr. Isaac Newton was sitting randomly under a tree. Suddenly, an apple fell on his head. He asked himself: "Oh, why do things fall downward on earth? why don't they go upward, to the right, or to the left?!" These thoughts have led him to discover the gravity. Gravity is the force that attracts things to each other. Without gravity, all of us would float in space, and there would be no life!"
We are not discussing the accuracy of this story here. However, the discovery of gravity in a general physical way, not in pure astronomical as it was before Newton, was a turning point in understanding the universe. Newton has stated the following statement, the **Newton's law of gravitation**, his treatise *Mathematical Principles of Natural Philosophy*

Definition 13.1. every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Assuming that the mass of the first particle is m_1, the mass of the second particle is m_2, and the distance between them is r, we can rewrite Newton's law of gravitation as following:

$$F_g = G\frac{m_1 m_2}{r^2} \tag{13.1}$$

were G is the *universal gravitational constant*:

$$G = 6.674 \times 10^{-11} N \cdot m^2/kg^2 \tag{13.2}$$

Remember the dimensional analysis.

> **Remark 13.1.** The distance r between the two particles is the distance between the centers of gravity (centers of mass) of the two particles.

For example, if the distance between the surfaces of the earth and the moon is d, there radii are r_e and r_m respectively, the gravitational force between them is:

$$F_g = G\frac{m_e m_m}{(r_e + r_m + d)^2} \tag{13.3}$$

> **Remark 13.2.** The gravitational force is negligible between two particles when the product of their masses is much less than G^{-1}. This is the reason we are not experiencing gravitational force between normal everyday objects like animals, trees, and cars.

Example 13.1

In an intense match between Liverpool and Chelsea, the ball, with 2 kg mass, is at $(0,0)$ and there are two players: the Chelsea player at $(4,4)$ with 97 kg mass and the Liverpool player at $(-2,3)$ with 80 kg mass. Find the resultant gravitational force at the ball.

Solution

Firstly, we must determine the distances between the ball and each one of the players (we are going to treat each one of them as points with zero radius):

$$r_c = \sqrt{(4-0)^2 + (4-0)^2} = 4\sqrt{2}$$
$$r_l = \sqrt{(0-(-2))^2 + (3-0)^2} = 5$$

Then, we will find the gravitational force between the ball and each of the players:

$$F_c = G\frac{m_b m_c}{r_c^2} = (6.674 \times 10^{-11})\frac{2 \times 97}{(4\sqrt{2})^2} = 4.046 \times 10^{-10} N$$

$$F_l = G\frac{m_b m_l}{r_l^2} = (6.674 \times 10^{-11})\frac{2 \times 80}{5^2} = 4.271 \times 10^{-10} N$$

We will resolute each of them on the x and y axis (review the vectors chapter):

$$F_x = F_c \cos\theta_1 - F_l \cos\theta_2 = \frac{4.046 \times 10^{-10}}{\sqrt{2}} - \frac{4.271 \times 10^{-10} \times 2}{\sqrt{13}} = 4.918 \times 10^{-11} N$$

$$F_y = F_c \sin\theta_1 + F_l \sin\theta_2 = \frac{4.046 \times 10^{-10}}{\sqrt{2}} + \frac{4.271 \times 10^{-10} \times 3}{\sqrt{13}} = 6.415 \times 10^{-10} N$$

Finally, we have to find the resultant:

$$F = \sqrt{F_x^2 + F_y^2} = \boxed{6.434 \times 10_{-10} N}$$

13.2 Gravitational potential energy

The potential energy is the type of energy that we have studied as a part of the mechanical energy. You may need to review the specific definition of particle-earth systems:

Definition 13.2. The potential energy is the energy stored in an object due to its height above the ground.

You may remember the law for potential energy ($U = mgh$) where h is the body's height, g is the earth's gravitational acceleration and m is the mass of the body. However, this law is valid only when the body is near enough to the earth so that the height effect on the gravitational acceleration is negligible. For a more generalized formula, we need to recall the law:

$$\Delta U = -\int_{r_i}^{r_f} F(r)dr$$

We then insert the formula for gravitational force:

$$\Delta U = -\int_{r_i}^{r_f} \frac{Gm_a m_b}{r^2}dr$$

Evaluate the integral:

$$\Delta U = -Gm_a m_b \int_{r_i}^{r_f} \frac{1}{r^2}dr = -Gm_a m_b \left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

Taking $r_i = \infty$ and $U_i = 0$, we get the final formula:

$$\boxed{U = -\frac{Gm_a m_b}{r}}$$

We can now propose a generalised definition for the gravitational potential energy.

Definition 13.3. The gravitational potential energy of a particle is the energy stored in a specific object due to its presence at specific distance in the gravitational field of another objects.

> **Remark 13.3.** The total gravitational energy of a system is the sum of the gravitational potential energies of all the pairs the objects in the system.

For example, if we have three objects with masses m_1, m_2 and m_3, we can find U_{total}:

$$U_{\text{total}} = U_{12} + U_{13} + U_{23} = -G\left(\frac{m_1 m_2}{r12} + \frac{m_1 m_3}{r13} + \frac{m_2 m_3}{r23}\right)$$

13.3 Orbits of planets and satellites

Recall the gravitational potential energy law that we have mentioned early in this chapter:

$$U = -\frac{GMm}{r}$$

Now, as we need to get a law for the total energy of the planet in orbit we have:

$$E = K + U = K - \frac{GMm}{r}$$

We can get the appropriate formula for the kinetic energy by making an equation of the gravitational force and the centripetal force:

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

Multiply both of them by r and divide by 2:

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$

We notice that the known expression for the kinetic energy $\frac{1}{2}mv^2$ has appeared. Hence, we can use the expression in the right-hand side for the energy equation:

$$E = \frac{GMm}{2R} - \frac{GMm}{R} = -\frac{GMm}{2r}$$

And this is for circular orbits. For elliptical orbits, we can use a instead of r:

$$E = -\frac{GMm}{2a}$$

We can see clearly that the energy of the planet is independent of any parameters but the semimajor axis and the masses of the orbiting planet and the orbited object.
Simple modification of the equation with the gravitational force and the centripetal force gives us the law for velocity of the planet in the orbit:

$$v = \sqrt{\frac{GM}{r}}$$

We are now in a position to calculate escape speed, which is the minimum speed the object must have at the Earth's surface to approach an infinite separation distance from the Earth. The law for the escape velocity is:

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

Where R is the radius of the earth and M is the earth's mass.

Example 13.2

A spacecraft is launched from earth and its mass is 7000 kg. Determine the kinetic energy it must have at the Earth's surface to move infinitely far away from the Earth.

Solution We can calculate the escape velocity using the law:

$$v_{esc} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2(6.67 \times 10^{-11} N \cdot m^2/kg^2)(5.97 \times 10^{24})}{(6.37 \times 10^6 m)}} = 1.12 \times 10^4 m/s$$

Then, we can use it in the kinetic energy law:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 7000 \times (1.12 \times 10^4)^2 = 4.39 \times 10^{11} J$$

Example 13.3

A geostationary satellite is a satellite which appears stationary in the space, as it's moving around the earth with a period equal to one revolution per day. Calculate its velocity.

Solution As we have the time period of the satellite (24 h= 86400 s), we can calculate its distance from the center of the earth using Kepler's third law:

$$r = \sqrt[3]{\frac{GM_E T^2}{4\pi^2}} = \sqrt[3]{\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})(86400)^2}{4\pi^2}} = 4.22 \times 10^7 m$$

We can then calculate its velocity with the law:

$$v = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})}{4.22 \times 10^7}} = 3.07 \times 10^3 m/s$$

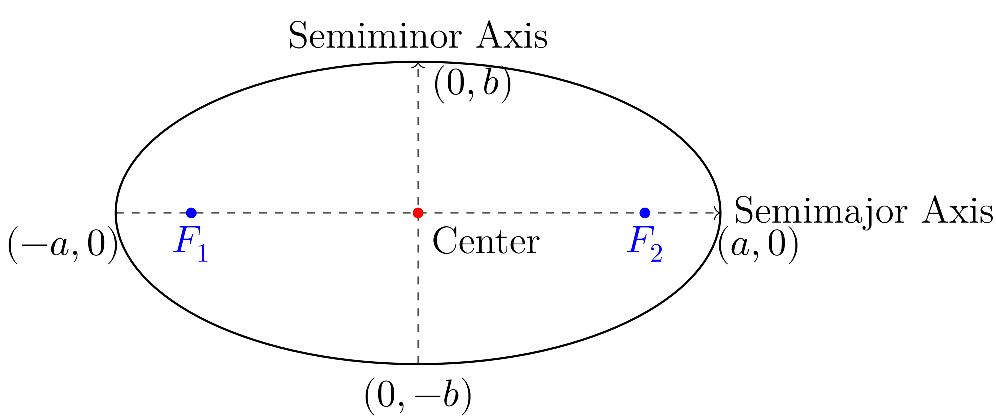
Geostationary satellites have several applications in real life. They are used to remain stationary above some-point in the earth to help in the GPS technology system.

13.4 Kepler's laws

Humans have been fascinated with planets and stars since their existence. The bright things in the sky were followed only by the naked eye until the invention of the telescope by a Dutch optician in 1608. However, for both curiosity and religious purposes, there were many models of how other bodies orbit inside and outside the solar system. Firstly, the *geocentric model* was developed by the Greek astronomer **Claudius Ptolemy** in the second century and was accepted for the following 14 centuries. The Danish astronomer **Tycho Brahe** collected data about 777 stars and planets in the sky in order to determine how the heavens were constructed. His assistant for a short period of time **Johannes Kepler**, used the data collected by Brahe and spent 16 years developing a mathematical model giving us his three laws of planetary motion:

> **Claim 13.1 — Kepler's first law of motion:** All planets move in elliptical orbits with the Sun at one focus.

Unlike the circular orbits hypothesized by early astronomers, Kepler showed that planets follow elliptical paths. An ellipse is defined by two foci, and the Sun occupies one of these foci. The varying distance between the planet and the Sun explains why planets move faster when closer to the Sun (perihelion) and slower when farther away (aphelion). The figure below shows the semi-minor and semi-major axes of the ellipse that models the planetary motion, along with the two foci.



Claim 13.2 — Kepler's second law of motion: The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

This law describes the speed of a planet along its orbit. When a planet is closer to the Sun (perihelion), it moves faster, and when it is farther away (aphelion), it moves slower. This phenomenon ensures that the area swept out by the line connecting the Sun and the planet remains constant over equal time intervals. We can express the change in the area caused by the planet over time as:

$$dA = \frac{1}{2}|r \times v|dt$$

As there is no external torque on the planet, the angular momentum (L) is constant:

$$\frac{dL}{dt} = 0$$

And from its definition:

$$|r \times v| = \frac{|L|}{m}$$

While m is the mass of the planet. From the two equations, we can conclude:

$$\frac{dA}{dt} = \frac{1}{2}\frac{|L|}{m}$$

As both of L and m are constants, the change in the area is constant.

Example 13.4

A comet orbits the Sun in an elliptical path. At its closest approach to the Sun (perihelion), the comet is $r_1 = 0.5\,\text{AU}$ (astronomical units) from the Sun and has a speed $v_1 = 40\,\text{km/s}$. At a later point in its orbit, when it is $r_2 = 2.0\,\text{AU}$ from the Sun (aphelion), what is its speed v_2? Assume the orbit obeys Kepler's Second Law.

Solution As the planet cuts equal areas in equal times, the following relationship is true:

$$v_1 r_2 = v_2 r_2$$

Hence:

$$v_2 = \frac{v_1 r_1}{r_2} = \frac{0.5 \times 40}{2} = 10\text{km/s}$$

Claim 13.3 — Kepler's third law of motion: The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

Kepler's third law can be predicted from the inverse-square law for circular orbits. Consider a planet of mass Mp that is assumed to be moving about the Sun mass M_S in a circular orbit. Playing a little bit with the geometry in the light of Newton's gravitation law we can obtain the result:

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3$$

We can approximate the radius in the law with the length of the semimajor axis, giving us the final law:

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) a^3$$

While T is the orbital orbit. The value $\frac{T^2}{a^3}$ is constant for every solar planet.

Example 13.5

The orbital radius of Mars is 1.52AU. If the orbital radius of earth is 1AU and its orbital period is one year, calculate the orbital period of mars.
Solution From Kepler's third law:

$$\frac{T_1^2}{a_1^3} = \frac{T_2^2}{a_2^3}$$

Hence:

$$T_2 = \sqrt{\frac{T_1^2 a_2^3}{a_1^3}} = \sqrt{\frac{1^2 \times 1.53^3}{1^3}} = 1.87 \text{ years}$$

Chapter 14

Fluid Mechanics

Contents

14.1 Pressure and Its Variation With Depth

Fluids, both liquids and gases, are substances that can flow and do not have a fixed shape. A key feature of fluids is that they do not sustain shearing stresses or tensile stresses. They only exert a stress that tends to compress the object submerged in the fluid from all sides. Their ability to exert force on objects immersed in them and on the walls of the container holding them is a consequence of the continuous collisions of their particles. This force is always perpendicular to the object's surface and is distributed over an area, resulting in pressure.

Definition 14.1. Pressure is the ratio of the force exerted on a submerged object to its surface area

$$P = \frac{F}{A} \tag{14.1}$$

Remark 14.1. 1. Pressure is a scalar quantity because it is the ratio of the component of the force normal to the area and it is independent on the size of the area

2. The unite of pressure in the SI system is Newton per square meter ($\frac{N}{m^2}$) or **Pascal** (Pa) where

$$1Pa = 1N/m^2 \tag{14.2}$$

Example 14.1

When a book 40 cm long, 30 cm wide, and 5 cm deep is submerged in water, a force of $F = 5N$ is exerted on all its sides. Find the ratio between the pressure on its biggest side to the pressure on its smallest side.

Solution. The pressure on the biggest side of the book would be: $P_b = \frac{F}{A} = \frac{5}{0.4 \cdot 0.3} = \frac{125}{3}$
The pressure on the smallest side of the book would be: $P_s = \frac{F}{A} = \frac{5}{0.3 \cdot 0.05} = \frac{1000}{3}$
Thus, the ratio between the two is $\frac{P_b}{P_s} = \frac{1}{8}$ ■

When dealing with fluids at rest, like water in a tank or air in the atmosphere, the pressure at any given point depends on the weight of the fluid above it. As you go deeper into a fluid, there is more fluid above you, which increases the pressure. This brings us to the concept of how pressure varies with depth in a fluid.

Claim 14.1 — The pressure at a specific depth within a fluid increases linearly with the depth. This can be explained by the fact that the deeper you go, the more weight of the fluid is "pressing down" on you. The mathematical relation for pressure at depth h under the surface of a fluid with density ρ is: ρ remains constant throughout. This assumption holds well for liquids like water but is less accurate for gases, which are compressible and experience changes in density with pressure.

$$P = P_0 + \rho g h \tag{14.3}$$

Where P_0 is the pressure at the surface of the fluid (usually the atmospheric pressure)

Remark 14.2. It is important to note that this equation assumes the fluid is in compressible, meaning its density ρ remains constant throughout. This assumption holds well for liquids like water but is less accurate for gases, which are compressible and experience changes in density with pressure.

Example 14.2

In The wide deep Atlantic, a submarine is at depth 25 m below the surface. What is the pressure exerted on the submarine?

Solution. Using water's density $\rho = 1000kg/m^3$, atmospheric pressure $P_0 = 101325Pa$, and $g = 9.8m/s^2$

$P = P_0 + \rho gh$

$P = 101325Pa + 1000kg/m^3 \cdot 9.8m/s^2 \cdot 25m = 346325Pa$ ■

14.1.1 Pascal's Law

The pressure in a fluid is influenced by both depth and the surface pressure (P_0). Any increase in pressure at the surface is transmitted throughout the entire fluid. This principle, known as **Pascal's law**, was first identified by the French scientist Blaise Pascal (1623–1662).

Claim 14.2 — Pascal's law states that a change in the pressure applied to a fluid is transmitted unchanged to every point of the fluid and the walls of the container.

An Important application to Pascal's principle is the hydraulic left. As shown in the figure below, a force of magnitude F_1 applied on the smaller piston of area A_1 could generate a much bigger force F_2 on the bigger piston of area A_2 according to the following law:

Claim 14.3 —

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2} \tag{14.4}$$

$$F_2 = F_1 \frac{A_2}{A_1} \tag{14.5}$$

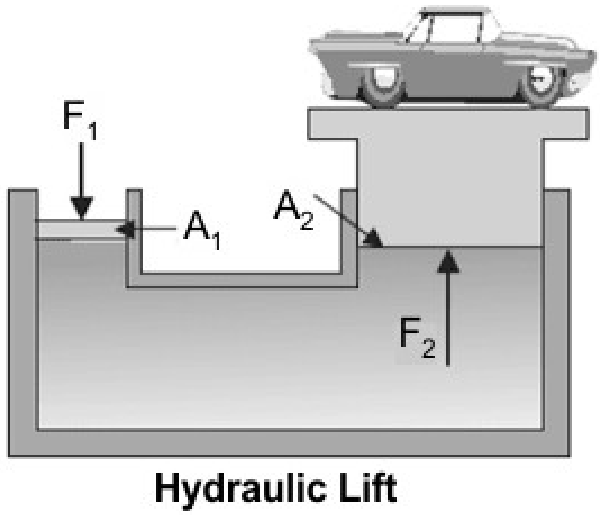


Figure 14.1: Pascal's law applied in a hydraulic lift

Example 14.3

Suppose the piston on the right side is 50 times wider than the piston on the left side. if a weight of $10N$ is placed on the lift piston, What is the upward force on the car on the right piston ?

Solution.

$$F_2 = F_1 \frac{A_2}{A_1} = 10 \cdot 50 = 500N \tag{14.6}$$

■

14.2 Buoyant Forces and Archimedes's Principle

When an object is immersed in a fluid, it experiences an upward force that opposes the weight of the object. This force, called the **buoyant force**, is responsible for the floating or sinking behavior of objects in fluids. Whether you're placing a small wooden block in water or watching a massive ship float, the principle governing these phenomena is universal.

Why do Buoyant Forces exist? You may ask. The reason objects experience an upward force when submerged in a fluid is due to the variation in pressure with depth. As we have seen earlier, pressure in a fluid increases with depth. Therefore, the pressure on the lower part of a submerged object is higher than the pressure on its upper part. This difference in pressure results in a net upward force, which is what we call the buoyant force.

Definition 14.2. The buoyant force is the upward force exerted by a fluid that opposes the weight of an object submerged in it.

Mathematically, we can describe the buoyant force as the difference in pressure across the submerged height of the object. The first one to recognize this principle was Archimedes, the ancient Greek mathematician and physicist, in his famous "Eureka! Eureka! " story. When the king asked Archimedes to validate the purity of the crown made for him by the blacksmith, the idea of testing the purity by sinking the crown in a tank came across the Greek innovator's mind. He noticed that he can measure the volume of the crown by the volume of displaced water and thus calculate its density to then compare it to pure gold and the story goes on. Archimedes also noticed an interesting relationship between the volume of displaced water and the buoyant force on the object, what is now known by **Archimedes principal**:

Definition 14.3. Archimedes principal states that any object, fully or partially submerged in a fluid, experiences a buoyant force equal to the weight of the fluid displaced by the object.

Claim 14.4 — For an object submerged in a fluid, the magnitude of the buoyant force F_b can be expressed as:

$$F_b = \rho_{fluid} g V_{submerged} \tag{14.7}$$

Where $V_{submerged}$ is the submerged volume of the object

Example 14.4

Consider a cube of side length $L = 1m$ submerged in water of density $\rho = 1000kg/m^3$ and $g = 9.8m/s^2$. Find the buoyant force on the cube when it is (a) fully submerged (b) half way in the water.

Solution. a

$$V_{submerged} = 1^3 = 1m^3 \tag{14.8}$$
$$F_b = 1000 \cdot 9.8 \cdot 1 = 9800N \tag{14.9}$$

b The $V_{submerged}$ in this case is only half the volume of the cube

$$F_b = 1000 \cdot 9.8 \cdot \frac{1}{2} = 4900N \tag{14.10}$$

■

Remark 14.3. The buoyant force does **NOT** depend on the weight or the material of the object itself but solely on the volume of fluid displaced and the density of the fluid. This is why a large steel ship can float, despite steel being much denser than water, while a steel nail would not float because it does not displace as much water.

Remark 14.4. A submerged object is in one of three states

1. If the buoyant force is greater than the object's weight, the object floats
2. If the buoyant force is less than the object's weight, the object sinks.
3. If the buoyant force equals the object's weight, the object remains suspended in the fluid.

14.3 Fluid Dynamics and Bernoulli's Equation

At its core, Fluid dynamics is the study of how fluids move, which is essential for understanding numerous natural as well as man-made phenomena. Unlike fluid statics, fluid dynamics is a rather complex topic. Fluids in motion exhibit a rather complex behaviour driven by pressure differences, flow speeds, and other factors. Before delving more into the equations, it is important to note that fluid dynamics usually assume ideal fluid flow

Definition 14.4. For an ideal fluid flow, we assume the following conditions

1. The fluid is incompressible: Its density is constant over time

2. The fluid is non-viscous: Internal fluid friction is neglected. An object moving in the fluid would have face no viscous friction

3. The flow is laminar: The velocity of the fluid at a point remains constant at any over time

4. The flow is irrational: The fluid have no angular momentum about any point

14.3.1 Continuity equation

The Continuity Equation expresses the conservation of mass in a fluid flow. It tells us that, for an incompressible fluid, the amount of fluid flowing into a region must be equal to the amount flowing out, ensuring mass conservation. Consider a fluid going into a pipe with a wide opening of A_1 with velocity v_1 and going out of a narrow ending with area A_2 and velocity v_2.

Claim 14.5 — The continuity equation for steady flow in a pipe is given by:

$$A_1 v_1 = A_2 v_2 \tag{14.11}$$

The equation shows that the product of the cross-sectional area and the fluid velocity is constant along the length of the pipe even if each element varies. If the area decreases (as in a constricted section of a pipe), the fluid must flow faster to maintain the same mass flow rate. This could be visualized when you pinch the end of a garden hose to increase the speed of water coming out.

Example 14.5

If the cross-sectional If the cross-sectional area of a pipe decreases from $A_1 = 0.04m^2$ to $A_2 = 0.01m^2$. What is the final velocity v_2 of the fluid passing through the narrower end if the initial velocity in the wider end is $v_1 = 2m/s$?

Solution. Using the continuity equation:

$$0.04 \cdot 2 = 0.01 \cdot v_2 \tag{14.12}$$

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14.3.2 Bernoulli's equation

While the continuity equation describes the conservation of mass in a flow, Bernoulli's equation describes the conservation of mechanical energy in a moving fluid. Consider a small volume of a fluid flowing through two points 1 and 2. At these points: The pressure is at point 1 is P_1 and at point 2 P_2, The cross-sectional areas are A_1 and A_2, the velocities are v_1 v_2, and the heights of these points to a reference level are h_1 and h_2.

The work done by the pressure at point 1 is the force times the distance. The force at point 1 is $F_1 = P_1 A_1$, so if the fluid moves a distance d_1, the work done becomes:

$$W = P_1 A_1 d_1 = P_1 V \tag{14.13}$$

When the same is applied to point 2 the net work would be:

$$W_{net} = \Delta PV \tag{14.14}$$

Since the mass of the fluid is $m = \rho V$, the kinetic energy becomes:

$$\Delta KE = \frac{1}{2}\rho V(v_2^2 - v_1^2) \tag{14.15}$$

Again using the mass substitution in the formula for potential energy $PE = mgh$, the potential energy becomes:

$$\Delta PE = \rho V g(h_2 - h_1) \tag{14.16}$$

Applying the work-energy theorem:

$$W_{net} = \Delta KE + \Delta PE \tag{14.17}$$

Substituting the expressions for work, kinetic energy, and potential energy, we get:

$$(P_2 - P_1)V = \frac{1}{2}\rho V(v_2^2 - v_1^2) + \rho V g(h_2 - h_1) \tag{14.18}$$

Simplifying by canceling the V:

$$P_2 - P_1 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) \tag{14.19}$$

Rearranging the terms:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 \tag{14.20}$$

Since the equation holds true for any two points in a streamline, the equation could be generalized

Claim 14.6 — Bernoulli's principle states that in a streamline flow, an increase in the velocity of the fluid results in a decrease in pressure or potential energy.

$$P + \frac{1}{2}\rho v^2 + \rho g h = constant \tag{14.21}$$